Er. L.K.Sharma an engineering graduate from NIT, Jaipur (Rajasthan), {Gold medalist, University of Rajasthan} is a well known name among the engineering aspirants for the last 15 years. He has been honored with BHAMASHAH AWARD two times for the academic excellence in the state of Rajasthan. He is popular among the student community for possessing the excellent ability to communicate the mathematical concepts in analytical and graphical way.

He has worked with many premiere IIT-JEE coaching institutes of Delhi, Haryana, Jaipur and Kota, {presently associated with THE GUIDANCE, Kalu Sarai, New Delhi as senior mathematics faculty and Head of Mathematics department with IGNEOUS, Sonipat (Haryana)}. He has worked with Delhi Public School, RK Puram, New Delhi for five years as a senior mathematics {IIT-JEE} faculty.

Er. L.K.Sharma has been proved a great supportive mentor for the last 15 years and the most successful IIT-JEE aspirants consider him an ideal mathematician for Olympiad/KVPY/ISI preparation. He is also involved in the field of online teaching to engineering aspirants and is associated with www.100marks.in and iProf Learning Sols India (P) Ltd for last 5 years , as a senior member of www.wiziq.com (an online teaching and learning portal), delivered many online lectures on different topics of mathematics at IIT-JEE {mains & advance} level.

# Contents

1. Basics of Mathematics ........................................ 1
2. Quadratic Equations ........................................ 14
3. Complex Numbers ........................................... 20
4. Binomial Theorem ............................................ 29
5. Permutation and Combination ............................ 32
6. Probability .................................................. 36
7. Matrices ...................................................... 42
8. Determinants ............................................... 55
9. Sequences and Series ....................................... 61
10. Functions .................................................... 67
11. Limits ......................................................... 76
12. Continuity and Differentiability ......................... 80
13. Differentiation ................................................ 86
14. Tangent and Normal ......................................... 91
15. Rolle's Theorem and Mean Value Theorem ............ 93
16. Monotonocity ............................................... 95
17. Maxima and Minima ......................................... 97
18. Indefinite Integral .......................................... 101
19. Definite Integral ............................................ 109
20. Area Bounded by Curves .................................. 114
21. Differential Equations ..................................... 118
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.</td>
<td>Basics of 2D-Geometry</td>
<td>123</td>
</tr>
<tr>
<td>23.</td>
<td>Straight Lines</td>
<td>125</td>
</tr>
<tr>
<td>24.</td>
<td>Pair of Straight Lines</td>
<td>129</td>
</tr>
<tr>
<td>25.</td>
<td>Circles</td>
<td>132</td>
</tr>
<tr>
<td>26.</td>
<td>Parabola</td>
<td>138</td>
</tr>
<tr>
<td>27.</td>
<td>Ellipse</td>
<td>142</td>
</tr>
<tr>
<td>28.</td>
<td>Hyperbola</td>
<td>146</td>
</tr>
<tr>
<td>29.</td>
<td>Vectors</td>
<td>152</td>
</tr>
<tr>
<td>30.</td>
<td>3-Dimensional Geometry</td>
<td>161</td>
</tr>
<tr>
<td>31.</td>
<td>Trigonometric Ratios and Identities</td>
<td>170</td>
</tr>
<tr>
<td>32.</td>
<td>Trigonometric Equations and Inequations</td>
<td>175</td>
</tr>
<tr>
<td>33.</td>
<td>Solution of Triangle</td>
<td>177</td>
</tr>
<tr>
<td>34.</td>
<td>Inverse Trigonometric Functions</td>
<td>180</td>
</tr>
</tbody>
</table>
1. BASICS of MATHEMATICS

{1} Number System :

(i) Natural Numbers
The counting numbers 1,2,3,4,..... are called Natural Numbers.
The set of natural numbers is denoted by N.
Thus N = {1,2,3,4, ......}.

(ii) Whole Numbers :
Natural numbers including zero are called whole numbers.
The set of whole numbers, is denoted by W.
Thus W = {0,1,2, .............}

(iii) Integers :
The numbers ... −3, −2, −1, 0, 1,2,3,....... are called integers and the set is denoted by I or Z.
Thus I (or Z) = {... −3, −2, −1, 0, 1, 2, 3...}

(a) Set of positive integers denoted by I⁺ and consists of {1,2,3,...} called as set of Natural numbers.
(b) Set of negative integers, denoted by I⁻ and consists of {..., −3, −2, −1}
(c) Set of non-negative integers {0,1,2 ......}, called as set of Whole numbers.
(d) Set of non-positive integers {..., −3, −2, −1,0}

(iv) Even Integers :
Integers which are divisible by 2 are called even integers.
e.g. 0, ±2, ±4.....

(v) Odd Integers :
Integers, which are not divisible by 2 are called as odd integers.
e.g. ±1, ±3, ±5, ±7......

(vi) Prime Number :
Let 'p' be a natural number, 'p' is said to be prime if it has exactly two distinct factors,
namely 1 and itself.
so, Natural number which are divisible by 1 and itself only are prime numbers (except 1).
e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

(vii) Composite Number :
Let 'a' be a natural number, 'a' is said to be composite if, it has at least three distinct factors.

Note:
(i) '1' is neither prime nor composite.
(ii) '2' is the only even prime number.
(iii) Number which are not prime are composite numbers (except 1).
(iv) '4' is the smallest composite number.

(viii) Co-prime Number :
Two natural numbers (not necessarily prime) are coprime, if there H.C.F (Highest common factor) is one.
e.g. (1,2), (1,3), (3,4), (3, 10), (3,8), (5,6), (7,8) etc.
These numbers are also called as **relatively prime** numbers.

**Note:**
(a) Two prime number(s) are always co-prime but converse need not be true.
(b) Consecutive numbers are always co-prime numbers.

(ix) **Twin Prime Numbers** :
If the difference between two prime numbers is two, then the numbers are twin prime numbers.

E.g. {3,5}, {5,7}, {11,13}, {17,19}, {29,31}

(x) **Rational Numbers** :
All the numbers that can be represented in the form \( p/q \), where \( p \) and \( q \) are integers and \( q \neq 0 \), are called rational numbers and their set is denoted by \( Q \).

Thus \( Q = \{\frac{p}{q} : p,q \in I \text{ and } q \neq 0\} \). It may be noted that every integer is a rational number since it can be written as \( p/1 \). It may be noted that all recurring decimals are rational numbers.

(xi) **Irrational Numbers** :
There are real numbers which cannot be expressed in \( p/q \) form. These numbers are called irrational numbers and their set is denoted by \( Q^c \). (i.e. complementary set of \( Q \)) e.g. \( \sqrt{2} \), \( 1 + \sqrt{3} \), \( e \), \( \pi \) etc. Irrational numbers cannot be expressed as recurring decimals.

**Note:** \( e \approx 2.71 \) is called Napier's constant and \( \pi \approx 3.14 \).

(xii) **Real Numbers** :
The complete set of rational and irrational number is the set of real numbers and is denoted by \( R \). Thus \( R = Q \cup Q^c \).

The real numbers can be represented as a position of a point on the real number line. The real number line is the number line where in the position of a point relative to the origin (i.e. 0) represents a unique real number and vice versa.

All the numbers defined so far follow the order property i.e. if there are two distinct numbers \( a \) and \( b \) then either \( a < b \) or \( a > b \).

**Note:**
(a) Integers are rational numbers, but converse need not be true.
(b) Negative of an irrational number is an irrational number.
(c) Sum of a rational number and an irrational number is always an irrational number e.g. \( 2 + \sqrt{3} \)
(d) The product of a non zero rational number & an irrational number will always be an irrational number.
(e) If \( a \in Q \) and \( b \not\in Q \), then \( ab = \) rational number, only if \( a = 0 \).
(f) Sum, difference, product and quotient of two irrational numbers need not be an irrational number or we can say, result may be a rational number also.
(xiii) **Complex Number :**
A number of the form \(a + ib\) is called complex number, where \(a, b \in \mathbb{R}\) and \(i = \sqrt{-1}\). Complex number is usually denoted by \(Z\) and the set of complex number is represented by \(C\).

**Note :** It may be noted that \(\mathbb{N} \subset \mathbb{W} \subset \mathbb{I} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}\).

{2} **Divisibility Test :**

(i) A number will be divisible by 2 iff the digit at the unit place of the number is divisible by 2.

(ii) A number will be divisible by 3 iff the sum of its digits of the number is divisible by 3.

(iii) A number will be divisible by 4 iff last two digit of the number together are divisible by 4.

(iv) A number will be divisible by 5 iff the digit of the number at the unit place is either 0 or 5.

(v) A number will be divisible by 6 iff the digits at the unit place of the number is divisible by 2 & sum of all digits of the number is divisible by 3.

(vi) A number will be divisible by 8 iff the last 3 digits of the number all together are divisible by 8.

(vii) A number will be divisible by 9 iff sum of all it's digits is divisible by 9.

(viii) A number will be divisible by 10 iff it's last digit is 0.

(ix) A number will be divisible by 11, iff the difference between the sum of the digits at even places and sum of the digits at odd places is 0 or multiple of 11.

e.g. 1298, 1221, 123321, 12344321, 1234554321, 123456654321

{3} (i) **Remainder Theorem :**
Let \(p(x)\) be any polynomial of degree greater than or equal to one and 'a' be any real number. If \(p(x)\) is divided by \((x - a)\), then the remainder is equal to \(p(a)\).

(ii) **Factor Theorem :**
Let \(p(x)\) be a polynomial of degree greater than or equal to 1 and 'a' be a real number such that \(p(a) = 0\), then \((x - a)\) is a factor of \(p(x)\). Conversely, if \((x - a)\) is a factor of \(p(x)\), then \(p(a) = 0\).

(iii) **Some Important Formulae :**

1. \((a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab\)
2. \((a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab\)
3. \(a^2 - b^2 = (a + b)(a - b)\)
4. \((a + b)^3 = a^3 + b^3 + 3ab(a + b)\)
5. \((a - b)^3 = a^3 - b^3 - 3ab(a - b)\)
6. \(a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)\)
7. \(a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)\)
8. \((a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\)
(9) \[ a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2} [(a-b)^2 + (b-c)^2 + (c-a)^2] \]
(10) \[ a^3 + b^3 + c^3 - 3abc = (a + b + c) (a^2 + b^2 + c^2 - ab - bc - ca) \]
\[ = \frac{1}{2} (a + b + c) [(a-b)^2 + (b-c)^2 + (c-a)^2] \]
(11) \[ a^4 - b^4 = (a + b)(a - b)(a^2 + b^2) \]
(12) \[ a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2) \]

{4} Definition of indices:

If 'a' is any non-zero real or imaginary number and 'm' is the positive integer, then \( a^m = a \cdot a \cdot a \cdot \ldots \cdot a \) (m times). Here a is called the base and m is the index, power or exponent.

(I) Law of indices:

(1) \( a^0 = 1 \), \( a \neq 0 \)

(2) \( a^{-m} = \frac{1}{a^m} \), \( a \neq 0 \)

(3) \( a^{m+n} = a^m \cdot a^n \), where m and n are rational numbers

(4) \( a^{m-n} = \frac{a^m}{a^n} \), where m and n are rational numbers, \( a \neq 0 \)

(5) \( (a^m)^n = a^{mn} \)

(6) \( a^{p/q} = \sqrt[q]{a^p} \)

{5} Ratio & proportion:

(i) Ratio:

1. If A and B be two quantities of the same kind, then their ratio is A : B; which may be denoted by the fraction \( \frac{A}{B} \) (This may be an integer or fraction )

2. A ratio may be represented in a number of ways e.g. \( \frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb} = \ldots \) where m, n, \ldots are non-zero numbers.

3. To compare two or more ratios, reduce them to common denominators.

4. Ratio between two ratios may be represented as the ratio of two integers e.g. \( \frac{a}{b} : \frac{c}{d} : \frac{a/b}{c/d} = \frac{ad}{bc} \) or \( ad : bc \) : duplicate, triplicate ratio.

5. Ratios are compounded by multiplying them together i.e. \( \frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = \frac{ace}{bdf} \ldots \)

6. If a : b is any ratio then its duplicate ratio is \( a^2 : b^2 \); triplicate ratio is \( a^3 : b^3 \ldots \) etc.

7. If a : b is any ratio, then its sub-duplicate ratio is \( a^{1/2} : b^{1/2} \); sub-triplicate ratio is \( a^{1/3} : b^{1/3} \) etc.
(ii) Proportion:
When two ratios are equal, then the four quantities compositing them are said to be proportional. If \( \frac{a}{b} = \frac{c}{d} \), then it is written as \( a : b = c : d \) or \( a : b :: c : d \)

1. 'a' and 'd' are known as extremes and 'b' and 'c' are known as means.
3. If \( a : b = c : d \), then \( b : a = d : c \) (Invertando)
4. If \( a : b = c : d \), then \( a : c = b : d \) (Alternando)
5. If \( a : b = c : d \), then \( \frac{a+b}{b} = \frac{c+d}{d} \) (Componendo)
6. If \( a : b = c : d \), then \( \frac{a-b}{b} = \frac{c+d}{d} \) (Dividendo)
7. If \( a : b = c : d \), then \( \frac{a+b}{b} = \frac{c+d}{c-d} \) (Componendo and Dividendo)

{6} Cross Multiplication:
If two equations containing three unknown are
\[
\begin{align*}
a_1x + b_1y + c_1z &= 0 \\
a_2x + b_2y + c_2z &= 0
\end{align*}
\]
Then by the rule of cross multiplication
\[
\begin{align*}
\frac{x}{b_1c_2 - b_2c_1} &= \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}
\end{align*}
\]
In order to write down the denominators of \( x, y \) and \( z \) in (3) apply the following rule, "write down the coefficients of \( x, y \) and \( z \) in order beginning with the coefficients of \( y \) and repeat them as in the diagram"

Multiply the coefficients across in the way indicated by the arrows; remembering that informing the products any one obtained by descending is positive and any one obtained by ascending is negative.

{7} Intervals:
Intervals are basically subsets of \( \mathbb{R} \) and are commonly used in solving inequalities or in finding domains. If there are two numbers \( a, b \in \mathbb{R} \) such that \( a < b \), we can define four types of intervals as follows:

<table>
<thead>
<tr>
<th>Symbols Used</th>
<th>Intervals</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Open interval : ( (a, b) ) = ( {x : a &lt; x &lt; b} ) i.e. end points are not included</td>
<td>( ( ) ) or ( [ ] )</td>
</tr>
<tr>
<td>(ii) Closed interval : ( [a, b] ) = ( {x : a \leq x \leq b} ) i.e. end points are also included</td>
<td>( [ ] )</td>
</tr>
<tr>
<td>This is possible only when both ( a ) and ( b ) are finite.</td>
<td></td>
</tr>
<tr>
<td>(iii) Open-closed interval : ( (a, b] ) = ( {x : a &lt; x \leq b} )</td>
<td>( ( ) ) or ( ] )</td>
</tr>
<tr>
<td>(iv) Closed-open interval : ( [a, b) ) = ( {x : a \leq x &lt; b} )</td>
<td>( [ ] ) or ( [ ] )</td>
</tr>
</tbody>
</table>
The infinite intervals are defined as follows:

(i) \((a, \infty) = \{x : x > a\}\)  
(ii) \([a, \infty) = \{x : x \geq a\}\)

(iii) \((-\infty, b) = \{x : x < b\}\)  
(iv) \((\infty, b] = \{x : x \leq b\}\)

(v) \((-\infty, \infty) = \{x : x \in \mathbb{R}\}\)

**Note:**

(i) For some particular values of \(x\), we use symbol \{\} e.g. If \(x = 1, 2\) we can write it as \(x \in \{1,2\}\)

(ii) If there is no value of \(x\), then we say \(x \in \emptyset\) (null set)

Various Types of Functions:

(i) **Polynomial Function:**
If a function \(f\) is defined by \(f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n\) where \(n\) is a non-negative integer and \(a_0, a_1, a_2, \ldots, a_n\) are real numbers and \(a_0 \neq 0\), then \(f\) is called a polynomial function of degree \(n\).

There are two polynomial functions, satisfying the relation; \(f(x).f(1/x) = f(x)+f(1/x)\), which are \(f(x) = 1 \pm x^n\)

(ii) **Constant function:**
A function \(f : A \rightarrow B\) is said to be a constant function, if every element of \(A\) has the same \(f\) image in \(B\). Thus \(f : A \rightarrow B; f(x) = c, \forall x \in A, c \in B\) is a constant function.

(iii) **Identity function:**
The function \(f : A \rightarrow A\) defined by, \(f(x) = x, \forall x \in A\) is called the identity function on \(A\) and is denoted by \(I_A\). It is easy to observe that identity function is a bijection.

(iv) **Algebraic Function:**
y is an algebraic function of \(x\), if it is a function that satisfies an algebraic equation of the form, \(P_0(x) y^n + P_1(x) y^{n-1} + \ldots + P_{n-1}(x) y + P_n(x) = 0\) where \(n\) is a positive integer and \(P_0(x), P_1(x), \ldots, P_n(x)\) are polynomials in \(x\), e.g. \(y = |x|\) is an algebraic function, since it satisfies the equation \(y^2 - x^2 = 0\).

All polynomial functions are algebraic but not the converse.

A function that is not algebraic is called Transcendental Function.

(v) **Rational Function:**
A rational function is a function of the form, \(y = f(x) = \frac{g(x)}{h(x)}\), where \(g(x) \& h(x)\) are polynomial functions.

(vi) **Irrational Function:**
An irrational function is a function \(y = f(x)\) in which the operations of additions, subtraction, multiplication, division and raising to a fractional power are used.

For example \(y = \frac{x^3 + x^{1/3}}{2x + \sqrt{x}}\) is an irrational function

(a) The equation \(\sqrt{f(x)} = g(x)\) is equivalent to the following system
\(f(x) = g^2(x) \& g(x) \geq 0\)

(b) The inequation \(\sqrt{f(x)} < g(x)\) is equivalent to the following system
\[ f(x) < g(x) \quad \& \quad f(x) \geq 0 \quad \& \quad g(x) > 0 \]

(c) The inequation \( \sqrt{f(x)} > g(x) \) is equivalent to the following system
\[ g(x) < 0 \quad \& \quad f(x) \geq 0 \quad \text{or} \quad g(x) \geq 0 \quad \& \quad f(x) > g^2(x) \]

(vii) Exponential Function:
A function \( f(x) = a^x = e^{x \ln a} \) \( (a > 0, \ a \neq 1, \ x \in \mathbb{R}) \) is called an exponential function. Graph of exponential function can be as follows:

\[
\begin{array}{c}
\text{Case - I} \\
\text{For } a > 1 \\
(0,1) \\
\end{array}
\]

\[
\begin{array}{c}
\text{Case - II} \\
\text{For } 0 < a < 1 \\
(0,1) \\
\end{array}
\]

(viii) Logarithmic Function:
\( f(x) = \log_a x \) is called logarithmic function where \( a > 0 \) and \( a \neq 1 \) and \( x > 0 \). Its graph can be as follows:

\[
\begin{array}{c}
\text{Case - I} \\
\text{For } a > 1 \\
(0,1) \\
\end{array}
\]

\[
\begin{array}{c}
\text{Case - II} \\
\text{For } 0 < a < 1 \\
(1,0) \\
\end{array}
\]

LOGARITHM OF A NUMBER:

The logarithm of the number \( N \) to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number \( N \). This number is designated as \( \log_a N \).

Hence: \[ \log_a N = x \iff a^x = N \quad , \quad a > 0 \quad , \quad a \neq 1 \quad \& \quad N > 0 \]

If \( a = 10 \), then we write \( \log b \) rather than \( \log_{10} b \).

If \( a = e \), we write \( \ln b \) rather than \( \log_e b \).

The existence and uniqueness of the number \( \log_a N \) follows from the properties of an experimental functions.

From the definition of the logarithm of the number \( N \) to the base 'a', we have an identity:
\[ a^{\log_a N} = N \quad , \quad a > 0 \quad , \quad a \neq 1 \quad \& \quad N > 0 \]

This is known as the Fundamental Logarithmic Identity.

Note: \( \log_a 1 = 0 \) \( (a > 0 \quad , \quad a \neq 1) \)

\[ \log_a a = 1 \quad (a > 0 \quad , \quad a \neq 1) \]

\[ \log_{1/a} a = -1 \quad (a > 0 \quad , \quad a \neq 1) \]
THE PRINCIPAL PROPERTIES OF LOGARITHMS:
Let \( M \) & \( N \) are arbitrary positive numbers, \( a > 0 \), \( a \neq 1 \), \( b > 0 \), \( b \neq 1 \) and \( \alpha \) is any real number then:

(i) \( \log_a (M \cdot N) = \log_a M + \log_a N \)

(ii) \( \log_a (M/N) = \log_a M - \log_a N \)

(iii) \( \log_a M^\alpha = \alpha \cdot \log_a M \)

(iv) \( \log_b M = \frac{\log_a M}{\log_a b} \)

Note: \( \log_b a \cdot \log_a b = 1 \leftrightarrow \log_b a = 1/\log_a b \).

\( \log_b a \cdot \log_c b \cdot \log_a c = 1 \)

\( e^{\ln a} = a \)

PROPERTIES OF MONOTONOCITY OF LOGARITHM:
(i) For \( a > 1 \) the inequality \( 0 < x < y \) & \( \log_a x < \log_a y \) are equivalent.

(ii) For \( 0 < a < 1 \) the inequality \( 0 < x < y \) & \( \log_a x > \log_a y \) are equivalent.

(iii) If \( a > 1 \) then \( \log_a x < p \Rightarrow 0 < x < a^p \)

(iv) If \( a > 1 \) then \( \log_a x > p \Rightarrow x > a^p \)

(v) If \( 0 < a < 1 \) then \( \log_a x < p \Rightarrow x > a^p \)

(vi) If \( 0 < a < 1 \) then \( \log_a x > p \Rightarrow 0 < x < a^p \)

\( \text{If the number & the base are on one side of the unity, then the logarithm is positive;} \)
\( \text{If the number & the base are on different sides of unity, then the logarithm is negative.} \)

\( \text{The base of the logarithm ‘} a \text{’ must not equal unity otherwise numbers not equal to unity will} \)
\( \text{not have a logarithm & any number will be the logarithm of unity.} \)

\( \text{For a non negative number ‘} a \text{’ & } n \geq 2 \text{, } n \in \mathbb{N} \quad \sqrt[n]{a} = a^{1/n}. \)

(ix) Absolute Value Function /Modulus Function:

The symbol of modulus function is \( f(x) =|x| \) and is defined as:
\[ y = |x|= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \]

Properties of Modulus:
For any \( a, b \in \mathbb{R} \)

(i) \( |a| \geq 0 \)

(ii) \( |a| = |−a| \)

(iii) \( |a| \geq a, \ |a| \geq −a \)

(iv) \( |ab| = |a| \cdot |b| \)

(v) \( |\frac{a}{b}| = \left|\frac{a}{b}\right| \)

(vi) \( |a + b| \leq |a| + |b| \)

(vii) \( |a − b| \geq ||a| − |b|| \)
(x) **Signum Function:**
A function $f(x) = \text{sgn}(x)$ is defined as follows:

$f(x) = \text{sgn}(x) = \begin{cases} 
1 & \text{for } x > 0 \\
0 & \text{for } x = 0 \\
-1 & \text{for } x < 0 
\end{cases}$

It is also written as $\text{sgn} x = \begin{cases} 
\left\lfloor \frac{x}{1} \right\rfloor, & x \neq 0 \\
0, & x = 0 
\end{cases}$

\[\text{sgn} f(x) = \begin{cases} 
\left\lfloor \frac{|f(x)|}{f(x)} \right\rfloor, & f(x) 
eq 0 \\
0, & f(x) = 0 
\end{cases}\]

(xi) **Greatest Integer Function or Step Up Function:**
The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ equals to the greatest integer less than or equal to $x$. For example:

- for $1 \leq x < 0$; $[x] = -1$; for $0 \leq x < 1$; $[x] = 0$
- for $1 \leq x < 2$; $[x] = 1$; for $2 \leq x < 3$; $[x] = 2$ and so on.

**Properties of greatest integer function:**

(a) $x - 1 < [x] \leq x$
(b) $[x + m] = [x] + m$ iff $m$ is an integer.
(c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$
(d) $[x] + [-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\
-1 & \text{otherwise} \end{cases}$

(xii) **Fractional Part Function:**
It is defined as, $y = \{x\} = x - [x]$.

E.g. the fractional part of the number 2.1 is $2.1 - 2 = 0.1$ and the fractional part of $-3.7$ is $0.3$.

The period of this function is 1 and graph of this function is as shown.

**Properties of fractional part function**

(a) $\{x + m\} = \{x\}$ iff $m$ is an integer
(b) $\{x\} + \{-x\} = \begin{cases} 0 & \text{if } x \text{ is an integer} \\
1 & \text{otherwise} \end{cases}$
Graphs of Trigonometric functions:

(a) \( y = \sin x \quad x \in \mathbb{R}; \quad y \in [-1,1] \)

(b) \( y = \cos x \quad x \in \mathbb{R}; \quad y \in [-1,1] \)

(c) \( y = \tan x \quad x \in \mathbb{R} - (2n+1)\pi/2; \quad n \in \mathbb{Z}; \quad y \in \mathbb{R} \)

(d) \( y = \cot x \quad x \in \mathbb{R} - n\pi; \quad n \in \mathbb{Z}; \quad y \in \mathbb{R} \)
Trigonometric Functions of sum or Difference of Two Angles :

(a) \( \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \)

(b) \( \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \)

(c) \( \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B) \cdot \sin(A-B) \)

(d) \( \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B) \cdot \cos(A-B) \)

(e) \( \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \)

(f) \( \cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A} \)

(g) \( \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} \)

Factorisation of the Sum or Difference of Two sines or cosines :

(a) \( \sin C + \sin D = 2\sin \frac{C+D}{2} \cos \frac{C-D}{2} \)

(b) \( \sin C - \sin D = 2\cos \frac{C+D}{2} \sin \frac{C-D}{2} \)

(c) \( \cos C + \cos D = 2\cos \frac{C+D}{2} \cos \frac{C-D}{2} \)

(d) \( \cos C - \cos D = -2\sin \frac{C+D}{2} \sin \frac{C-D}{2} \)

Transformation of Products into Sum or Differences of Sines & Cosines :

(a) \( 2\sin A \cos B = \sin(A+B) + \sin(A-B) \)

(b) \( 2\cos A \sin B = \sin(A+B) - \sin(A-B) \)

(c) \( 2\cos A \cos B = \cos(A+B) + \cos(A-B) \)

(d) \( 2\sin A \sin B = \cos(A-B) - \cos(A+B) \)
Multiple and Sub-multiple Angles:

(a) \( \sin 2A = 2 \sin A \cos A \); \( \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \)

(b) \( \cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A; \) \( 2 \cos^2 \frac{\theta}{2} = 1 + \cos \theta, 2 \sin^2 \frac{\theta}{2} = 1 - \cos \theta \).

(c) \( \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \); \( \tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}} \)

(d) \( \sin 2A = \frac{2 \tan A}{1 + \tan^2 A} \); \( \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} \)

(e) \( \sin 3A = 3 \sin A - 4 \sin^3 A \)

(f) \( \cos 3A = 4 \cos^3 A - 3 \cos A \)

(g) \( \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} \)

Important Trigonometric Ratios:

(a) \( \sin n \pi = 0 \); \( \cos n \pi = (-1)^n \); \( \tan n \pi = 0 \), where \( n \epsilon \mathbb{I} \)

(b) \( \sin 15^\circ \) or \( \sin \frac{\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \cos 75^\circ \) or \( \cos \frac{5\pi}{12} \)

\( \cos 15^\circ \) or \( \cos \frac{\pi}{12} = \frac{\sqrt{3} + 1}{2\sqrt{2}} = \sin 75^\circ \) or \( \sin \frac{5\pi}{12} \)

\( \tan 15^\circ = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3} = \cot 75^\circ \); \( \tan 75^\circ = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3} \) or \( \cot 15^\circ \)

(c) \( \sin \frac{\pi}{10} \) or \( \sin 18^\circ = \frac{\sqrt{5} - 1}{4} \) & \( \cos 36^\circ \) or \( \cos \frac{\pi}{5} = \frac{\sqrt{5} + 1}{4} \)

Conditional Identities:

If \( A + B + C = \pi \) then:

(i) \( \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C \)

(ii) \( \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \)

(iii) \( \cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C \)

(iv) \( \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \)

(v) \( \tan A + \tan B + \tan C = \tan A \tan B \tan C \)

(vi) \( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{A}{2} = 1 \)

(vii) \( \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \)

(viii) \( \cot A \cot B + \cot B \cot C + \cot C \cot A = 1 \)
(ix) \( A + B + C = \frac{\pi}{2} \) then \( \tan A \tan B + \tan B \tan C + \tan C \tan A = 1 \)

**Range of Trigonometric Expression :**

\[
E = a \sin \theta + b \cos \theta
\]

\[
E = \sqrt{a^2 + b^2} \sin(\theta + \alpha), \text{ where } \tan \alpha = \frac{b}{a}
\]

\[
= \sqrt{a^2 + b^2} \cos(\theta - \beta), \text{ where } \tan \beta = \frac{a}{b}
\]

Hence for any real value of \( \theta \), \(-\sqrt{a^2 + b^2} \leq E \leq \sqrt{a^2 + b^2}\)

**Sine and Cosine Series :**

\[
\sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin(\alpha + (n-1)\beta) = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \sin \left(\alpha + \frac{n-1}{2} \beta\right)
\]

\[
\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + (n-1)\beta) = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos \left(\alpha + \frac{n-1}{2} \beta\right)
\]
2. QUADRATIC EQUATION

If $a_0 \neq 0$, polynomial equation of 'n' degree is represented by $a_n x^n + a_{n-1} x^{n-1} + a_2 x^2 + \ldots + a_2 = 0$, where $n \in N$. If $n = 1$, equation is termed as 'linear' and if $n=2$, equation is termed as quadratic equation.

**Note:**
(i) Values of 'x' which satisfy the polynomial equation is termed as its roots or zeros.
(ii) If general, polynomial equation of 'n' degree is having n roots, but if it is having more than n roots, then it represents an identity.

For example: 
\[
\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1
\]
is quadratic equation but have more than two roots (viz: $x = a, b, c, \ldots$ etc.)
(iii) If $ax^2 + bx + c = 0$ is an identity, then $a = b = c = 0$.
(iv) An identity is satisfied for all real values of the variable.

For example: $\sin^2 x + \cos^2 x = 1 \quad \forall \ x \in \mathbb{R}$.

2. Relation Between Roots and Coefficients:

(i) The solutions of quadratic equation $ax^2 + bx + c = 0$, ($a \neq 0$) is given by

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } b^2 - 4ac = D \text{ is called discriminant of quadratic equation.}
\]

(ii) If $\alpha, \beta$ are the roots of quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, then:

$$ax^2 + bx + c = a(x - \alpha)(x - \beta) \Rightarrow$$

- $\alpha + \beta = \frac{-b}{a}$
- $\alpha \beta = \frac{c}{a}$
- $|\alpha - \beta| = \frac{\sqrt{D}}{|a|}$

(iii) A quadratic equation whose roots are $\alpha$ and $\beta$ is $(x - \alpha)(x - \beta) = 0$

$$\Rightarrow x^2 - \{\text{sum of roots}\} x + \{\text{product of roots}\} = 0$$

**Note:**
- If $\alpha, \beta, \gamma$ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

\[
S_1 = \alpha + \beta + \gamma = -\frac{b}{a},\quad S_2 = \beta \gamma + \alpha \beta = \frac{c}{a} \quad \text{and} \quad S_3 = \alpha \beta \gamma = -\frac{d}{a}; \quad \text{(where $S_k$ represents sum of roots taking k roots at a time)}
\]

- If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

\[
\alpha + \beta + \gamma + \delta = -\frac{b}{a},\quad (\alpha + \beta)(\gamma + \delta) + \alpha \beta + \gamma \delta = \frac{c}{a},\quad \alpha \beta (\gamma + \delta) + \gamma \delta (\alpha + \beta) = -\frac{d}{a},\quad \alpha \beta \gamma \delta = \frac{e}{a}.
\]

- If $a_1, a_2, a_3, \ldots, a_n$ are roots of the equation $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad \ldots\ldots\ldots\ldots(1)$

then, $f(x) = a_n (x - a_1)(x - a_2)\ldots\ldots\ldots\ldots(x - a_n)$

$\therefore a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = a_n (x - a_1)(x - a_2)\ldots\ldots\ldots\ldots(x - a_n)$

Comparing the coefficients of like powers of $x$ on both sides, we get

\[
S_1 = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n = -\frac{a_{n-1}}{a_n} = \frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}
\]
3. **Nature of Roots**:

Consider the quadratic equation $ax^2 + bx + c = 0$ having $\alpha, \beta$ as its roots, $D=b^2-4ac$

\[ \begin{align*}
\text{Nature of Roots} & \\
D & = 0 \\
(\text{Roots are equal, } \alpha = \beta = -b/2a) & \\
D & \neq 0 \\
(\text{Roots are unequal}) & \\
\end{align*} \]

\[ \begin{align*}
a, b, c & \in \mathbb{R} \text{ and } D > 0 \\
& \Rightarrow \text{Roots are real and unequal} \\
a, b, c & \in \mathbb{R} \text{ and } D < 0 \\
& \Rightarrow \text{Roots are imaginary} \\
\end{align*} \]

\[ \begin{align*}
a, b, c & \in \mathbb{Q} \text{ and } D \text{ is a perfect square} \\
& \Rightarrow \text{Roots are rational} \\
a, b, c & \in \mathbb{Q} \text{ and } D \text{ is not a perfect square} \\
& \Rightarrow \text{Roots are irrational} \\
\end{align*} \]

\[ \begin{align*}
a = 1, b, c & \in \mathbb{I} \text{ and } D \text{ is a perfect square} \\
& \Rightarrow \text{Roots are integral.} \\
\end{align*} \]

**Note:**

(i) If the coefficients of the equation $ax^2 + bx + c = 0$ are all real and $\alpha + \beta \text{ is its root, then } \alpha - \beta \text{ is also a root. i.e. imaginary roots occur in conjugate pairs.}$

(ii) If the coefficients in the equation are all rational and $\alpha + \beta \sqrt{\gamma}$ one of its roots, then $\alpha - \beta \sqrt{\gamma}$ is also a root where $\alpha, \beta \in \mathbb{Q}$ and $\beta$ is not a perfect square.

(iii) If a quadratic expression $f(x) = ax^2 + bx + c$ is a perfect square of a linear expression then $D=b^2-4ac = 0$. 
4. Common Roots:

Consider two quadratic equations, \( a_1x^2 + b_1x + c_1 = 0 \) and \( a_2x^2 + b_2x + c_2 = 0 \).

(i) If two quadratic equations have both roots common, then the equations are identical and their co-efficient are in proportion. i.e.

\[
\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}
\]

(ii) If only one root is common, then the common root \( '\alpha' \) will be:

\[
\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}
\]

Hence the condition for one common root is

\[
(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2
\]

Note:

(i) If \( f(x) = 0 \) and \( g(x) = 0 \) are two polynomial equation having some common root(s) then these common root(s) is/are also the root(s) of \( h(x) = af(x) + bg(x) = 0 \).

(ii) To obtain the common root, make coefficients of \( x^2 \) in both the equations same and subtract one equation from the other to obtain a linear equation in \( x \) and then solve it for \( x \) to get the common root.

5. Factorisation of Quadratic Expressions:

If a quadratic expression \( f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c \) may be resolved into two linear factors, then

\[
\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0
\]

6. Graph of Quadratic Expression:

\[
y = f(x) = ax^2 + bx + c; \quad a \neq 0
\]

\[
\Rightarrow \quad \left(y + \frac{D}{4a}\right) = a \left(x + \frac{b}{2a}\right)^2
\]

Note:

(i) \( f(x) = ax^2 + bx + c \) represents a parabola.

(ii) the co-ordinate of vertex are \( \left(\frac{-b}{2a}, \frac{-D}{4a}\right) \)

(iii) If \( a > 0 \) then the shape of the parabola is concave upwards and if \( a < 0 \) then the shape of the parabola is concave downwards.

(iv) the parabola intersect the y-axis at point \((0,c)\).

(v) the x-co-ordinate of point of intersection of parabola with x-axis are the real roots of the quadratic equation \( f(x) = 0 \).
7. Range of Quadratic Expression \( f(x) = ax^2 + bx + c \).

(1) Absolute Range :

If \( a > 0 \) \( \Rightarrow \) \( f(x) \in \left[ -\frac{D}{4a}, \infty \right) \) and if \( a < 0 \) \( \Rightarrow \) \( f(x) \in \left( -\infty, -\frac{D}{4a} \right] \).

Hence maximum and minimum values of the expression \( f(x) \) is \(-\frac{D}{4a}\) in respective cases and it occurs at \( x = -\frac{b}{2a} \) (at vertex).

(2) Range in restricted domain :

(a) If \(-\frac{b}{2a} \not\in [x_1, x_2] \), then \( f(x) \in \{ \min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\} \} \)

(b) If \(-\frac{b}{2a} \in [x_1, x_2] \), then \( f(x) \in \{ \min\{f(x_1), f(x_2), -\frac{D}{4a}\}, \max\{f(x_1), f(x_2), -\frac{D}{4a}\} \} \)

8. Sign of quadratic Expressions :

The value of expression \( f(x) = ax^2 + bx + c \) at \( x = x_0 \) is equal to y-co-ordinate of a point on parabola \( y = ax^2 + bx + c \) whose x-co-ordinate is \( x_0 \). Hence if the point lies above the x-axis for some \( x = x_0 \) then \( f(x_0) > 0 \) as illustrated in following graphs:

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Note:

(i) \( ax^2 + bx + c \geq 0 \) \( \forall \ x \in \mathbb{R} \) \( \Rightarrow \) \( a > 0 \) and \( D \leq 0 \).

(ii) \( ax^2 + bx + c \leq 0 \) \( \forall \ x \in \mathbb{R} \) \( \Rightarrow \) \( a < 0 \) and \( D \leq 0 \).

(iii) \( ax^2 + bx + c > 0 \) \( \forall \ x \in \mathbb{R} \) \( \Rightarrow \) \( a > 0 \) and \( D < 0 \).

(iv) \( ax^2 + bx + c < 0 \) \( \forall \ x \in \mathbb{R} \) \( \Rightarrow \) \( a < 0 \) and \( D < 0 \).
10. Location of Roots:

Let \( f(x) = ax^2 + bx + c \), where \( a, b, c \in \mathbb{R} \).

(A) \( (x_0, f(x_0)) \)

(B) \( (x_0, f(x_0)) \)

(C) \( (x_0, f(x_0)) \)

(i) Conditions for both the roots of \( f(x) = 0 \) to be greater than a specified number \( 'x_0' \) are \( b^2 - 4ac \geq 0 \); \( af(x_0) > 0 \) and \( (-b/2a) > x_0 \). (refer figure no. (A))

(ii) Conditions for both the roots of \( f(x) = 0 \) to be smaller than a specified number \( 'x_0' \) are \( b^2 - 4ac \geq 0 \); \( af(x_0) > 0 \) and \( (-b/2a) < x_0 \). (refer figure no. (B))

(iii) Conditions for both roots of \( f(x) = 0 \) to lie on either side of the number \( 'x_0' \) (in other words the number \( 'x_0' \) lies between the roots of \( f(x) = 0 \)) is \( b^2 - 4ac > 0 \) and \( af(x_0) < 0 \). (refer figure no. (C))

(D) \( (x_1, f(x_1)) \) and \( (x_2, f(x_2)) \)

(E) \( (x_1, f(x_1)) \) and \( (x_2, f(x_2)) \)

(iv) Condition that both roots of \( f(x) = 0 \) to be confined between the numbers \( x_1 \) and \( x_2 \), \( x_1 < x_2 \) are \( b^2 - 4ac \geq 0 \); \( af(x_1) > 0 \); \( af(x_2) > 0 \) and \( x_1 < (-b/2a) \) \(< x_2 \). (refer figure no. (D))

(v) Conditions for exactly one root of \( f(x) = 0 \) to lie in the interval \((x_1, x_2)\) i.e. \( x_1 < \alpha < x_2 \) is \( f(x_1)f(x_2) < 0 \). (refer figure no. (E))

Important Results

(i) If \( \alpha \) is a root of the equation \( f(x) = 0 \), then the polynomial \( f(x) \) is divisible by \( (x - \alpha) \) or \( (x - \alpha) \) is a factor of \( f(x) \)

(ii) An equation of odd degree will have odd number of real roots and an equation of even degree will have even numbers of real roots.

(iii) Every equation \( f(x) = 0 \) of odd degree has at least one real root of a sign opposite to that of its last constant term. (If coefficient of highest degree term is positive).

(iv) The quadratic equation \( f(x) = ax^2 + bx + c = 0 \), \( a \neq 0 \) has \( \alpha \) as a repeated root if and only if \( f(\alpha) = 0 \) and \( f'(\alpha) = 0 \). In this case \( f(x) = a(x - \alpha)^2 \Rightarrow \alpha = -b/2a \).

(v) If polynomial equation \( f(x) = 0 \) has a root \( \alpha \) of multiplicity \( r \) (where \( r > 1 \)), then \( f(x) \) can be written as \( f(x) = (x - \alpha)^r g(x) \), where \( g(\alpha) \neq 0 \), also, \( f(x) = 0 \) has \( \alpha \) as a root with multiplicity \( r-1 \).
(vi) If polynomial equation \( f(x) = 0 \) has \( n \) distinct real roots, then
\[
f(x) = a_0(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n),
\]
where \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are \( n \) distinct real roots and
\[
\frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{1}{x - \alpha_k}
\]

(vii) If there be any two real numbers 'a' and 'b' such that \( f(a) \) and \( f(b) \) are of opposite signs, then \( f(x) = 0 \) must have odd number of real roots (also at least one real root) between 'a' and 'b'

(viii) If there be any two real numbers 'a' and 'b' such that \( f(a) \) and \( f(b) \) are of same signs, then \( f(x) = 0 \) must have even number of real roots between 'a' and 'b'

(ix) If polynomial equation \( f(x) = 0 \) has \( n \) real roots, then \( f'(x) = 0 \) has at least \( (n - 1) \) real roots.

(x) **Descartes rule** of signs for the roots of a polynomial

- The maximum number of positive real roots of a polynomial equation \( f(x) = 0 \) is the number of changes of the signs of coefficients of \( f(x) \) from positive to negative or negative to positive.
- The maximum number of negative real roots of the polynomial equation \( f(x) = 0 \) is the number of changes from positive to negative or negative to positive in the signs of coefficients of \( f(-x) = 0 \).

(xi) To form an equation whose roots are reciprocals of the roots in equation
\[
a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 , \ x \text{ is replaced by } 1/x \text{ and then both sides are multiplied by } x^n.
\]
3. COMPLEX NUMBER

(1) Complex number system:
A number of the form \( a + ib \) where \( a, b \in \mathbb{R} \), and \( i = \sqrt{-1} \) is called a complex number. If \( z = a + ib \), then the real part of \( z \) is denoted by \( \text{Re}(z) \) and the imaginary part of \( z \) is denoted by \( \text{Im}(z) \). A complex number \( z \) is said to be purely real if \( \text{Im}(z) = 0 \) and is said to be purely imaginary if \( \text{Re}(z) = 0 \).

Note:
- Set of all complex numbers is denoted by \( \mathbb{C} \), where \( \mathbb{C} = \{a + ib : b \in \mathbb{R} \land i = \sqrt{-1} \} \)
- Zero is the only number which is purely real as well as purely imaginary.
- \( i = \sqrt{-1} \) is the imaginary unit and is termed as 'iota'.
- If \( n \in \mathbb{I} \), then \((i)^{4n} = 1\), \((i)^{4n+1} = i\), \((i)^{4n+2} = -1\) and \((i)^{4n+3} = -i\)
- Sum of four consecutive powers of 'i' is always zero (i.e. \((i)^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3} = 0\) ), where \( n \in \mathbb{I} \).
- The property of multiplication \((\sqrt{a})(\sqrt{b}) = \sqrt{ab} \in \) is valid only if atleast one of a or b is non-negative.

(2) Algebraic Operations:

1. Addition : \((a + bi) + (c + di) = (a + c) + (b + d)i\)
2. Subtraction : \((a + bi) - (c + di) = (a - c) + (b - d)i\)
3. Multiplication : \((a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i\)
4. Division : \(\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \left(\frac{c-di}{c-di}\right) = \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i\)

Note:
- (i) In real numbers if \( a^2 + b^2 = 0 \) then \( a = b = 0 \) but in complex numbers, \( z_1^2 + z_2^2 = 0 \) does not imply \( z_1 = z_2 = 0 \).
- (ii) Inequalities in complex numbers are not defined (Law of order is not applicable for complex numbers).

(3) Equality In Complex Number:
Two complex numbers \( z_1 = a_1 + ib_1 \) and \( z_2 = a_2 + ib_2 \) are equal if and only if their corresponding real and imaginary parts are equal respectively
\[ z_1 = z_2 \Rightarrow \text{Re}(z_1) = \text{Re}(z_2) \quad \text{and} \quad \text{Im}(z_1) = \text{Im}(z_2). \]

(4) Conjugate of Complex Number:
Conjugate of a complex number \( z = a + ib \) is denoted by \( \bar{z} \) and is defined as \( \bar{z} = a - ib \). Geometrically a complex number \( z = a + ib \) is represented by ordered pair \((a, b)\) on the complex plane (or Argand plane) and the mirror image of \( z \) about real axis represents the conjugate of \( z \).
Properties of Conjugate of a Complex Number:

- $z_1 = z_2 \iff \overline{z_1} = \overline{z_2}$
- $z + \overline{z} = 2 \text{Re}(z)$
- $z = \overline{z} \iff z$ is purely real
- $z\overline{z} = |z|^2$ if $z \neq 0$
- $\frac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$
- If $f(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n$ where $a_0, a_1, \ldots, a_n$ and $z$ are complex number, then
  $f(\overline{z}) = \overline{a_0} + \overline{a_1}(\overline{z}) + \overline{a_2}(\overline{z})^2 + \ldots + \overline{a_n}(\overline{z})^n$

For example: $\frac{2z + 3z^3}{4z + 1} = \frac{2\overline{z} + 3(\overline{z})^3}{4\overline{z} + 1}$

(5) Modulus of a Complex Number:

If $z = a + ib$, then modulus of complex number $z$ is denoted by $|z|$ and defined as $|z| = \sqrt{a^2 + b^2}$. Geometrically $|z|$ is the distance of $z$ from origin on the complex plane.

Properties of modulus:

- $|z| = 0 \iff z = 0$
- $-|z| \leq \text{Re}(z) \leq |z|$
- $z\overline{z} = |z|^2$
- $|z_1z_2| = |z_1||z_2|$
\[ \frac{|z_1|}{|z_2|} = |\frac{z_1}{z_2}|, \text{ if } z_2 \neq 0 \]

\[ |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1z_2 - z_1\bar{z}_2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2) \]

\[ |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2) \]

- If \( z_1, z_2 \neq 0 \), then \( |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2} \text{ is purely imaginary} \).
- Geometrically \( |z_1 - z_2| \) represents the distance between complex points \( z_1 \) and \( z_2 \) on argand plane.

**Triangle Inequality :**

If \( z_1 \) and \( z_2 \) are two complex numbers, then

- \( |z_1 + z_2| \leq |z_1| + |z_2| \).
- \( |z_1 - z_2| \geq ||z_1| - |z_2|| \)

The sign of equality holds iff \( z_1, z_2 \) and origin are collinear and \( z_1, z_2 \) lies on same side of origin.

(6) **Argument of a Complex Number :**

If non-zero complex number \( z \) is represented on the argand plane by complex point \( 'P' \), then argument (or amplitude) of \( z \) is the angle which \( OP \) makes with positive direction of real axis.

Let complex number \( z = a + ib \) and \( \alpha = \tan^{-1}\left|\frac{b}{a}\right| \), then principal argument of \( z \) is given as

- \( \arg(z) = \alpha \), if \( z \) lies in I quadrant.
- \( \arg(z) = (\pi - \alpha) \), if \( z \) lies in II quadrant.
- \( \arg(z) = (-\pi + \alpha) \), if \( z \) lies in III quadrant.
- \( \arg(z) = -\alpha \), if \( z \) lies in IV quadrant.

**Note:**

- Argument of a complex number is not unique, according to definition if \( \theta \) is a value of argument, then \( 2\pi n + \theta \ \forall \ n \in \mathbb{I} \) is also the argument of complex number.
- If \( \theta \) is argument of complex number and \( \theta \in (-\pi, \pi] \), then the argument \( \theta \) is termed as principal argument.
- Unless otherwise mentioned, \( \arg(z) \) implies the principal argument.

**Properties of Argument of Complex Number :**

- \( \arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2k\pi \) for some integer \( k \).
- \( \arg(z_1/z_2) = \arg(z_1) - \arg(z_2) + 2k\pi \) for some integer \( k \).
- \( \arg(z^n) = n \arg(z) + 2k\pi \) for some integer \( k \).
- \( \arg(z) = 0 \Leftrightarrow z \) is real, for any complex number \( z \neq 0 \)
- \( \arg(z) = \pm \pi/2 \Leftrightarrow z \) is purely imaginary, for any complex number \( z \neq 0 \)
- \( \arg(\bar{z}) = -\arg(z) \)
(7) **Representations Of A Complex Number:**

(a) **Cartesian Form (Geometric Representation):**

Complex number \( z = x + iy \) is represented by a point on the complex plane (Argand plane/Gaussian plane) by the ordered pair \((x, y)\).

(b) **Trigonometric/Polar Representation:**

\[ z = r \cos \theta + ir \sin \theta, \] \( |z| = r \) and \( \arg z = \theta \), represents the polar form of complex number

**Note:**

- \( \cos \theta + i \sin \theta \) is also written as \( \text{cis} \theta \) or \( e^{i\theta} \).
- \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \) and \( \sin x = \frac{e^{ix} - e^{-ix}}{2i} \) are known as Euler's Identities.

(c) **Euler's Representation:**

\[ z = re^{i\theta}, \] \( |z| = r \) and \( \arg z = \theta \), represents the Euler's form of complex number.

(d) **Vectorial Representation:**

Every complex number can be considered as if it is the position vector of a point. If the point \( P \) represents the complex number \( z \) then, \( \overrightarrow{OP} = z \) and \( |\overrightarrow{OP}| = |z| \).

(8) **De Moivre's Theorem:**

- If \( n \) is any rational number, then \( (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \).

- If \( z = (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)(\cos \theta_3 + i \sin \theta_3) \ldots (\cos \theta_n + i \sin \theta_n) \) then
\[
z = \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i \sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n), \] where \( \theta_1, \theta_2, \theta_3 \ldots, \theta_n \in \mathbb{R} \)

- If \( z = r(\cos \theta + i \sin \theta) \) and \( n \) is a positive integer, then
\[
z^{1/n} = r^{1/n} \left[ \cos \left(\frac{2k\pi + \theta}{n}\right) + i \sin \left(\frac{2k\pi + \theta}{n}\right) \right], \] where \( k = 0, 1, 2, 3, \ldots, (n-1) \)

- If \( p, q \in \mathbb{Z} \) and \( q \neq 0 \), then \( (\cos \theta + i \sin \theta)^{p/q} = \cos \left(\frac{2k\pi + p\theta}{q}\right) + i \sin \left(\frac{2k\pi + p\theta}{q}\right) \),
where \( k = 0, 1, 2, 3, \ldots, (q-1) \).

**Note:**

This theorem is not valid when \( n \) is not a rational number or the complex number is not in the form of \( \cos \theta + i \sin \theta \).
(9) **Cube Root Of Unity:**

Let \( x = (1)^{1/3} \)
\[
\Rightarrow x^3 - 1 = 0 \quad \Rightarrow (x - 1)(x^2 + x + 1) = 0.
\]
\[
\Rightarrow x = 1, \quad \frac{-1 + i\sqrt{3}}{2}, \quad \frac{-1 - i\sqrt{3}}{2}
\]
If \( \omega = \frac{-1 + i\sqrt{3}}{2} \), then \( \omega^2 = \frac{-1 - i\sqrt{3}}{2} \)

\[ \therefore \text{Cube Roots of unity is given by: } 1, \omega, \omega^2. \]

**Properties of cube roots of unity:**

- \( \omega^{3n} = 1; \omega^{3n+1} = \omega; \omega^{3n+2} = \omega^2 \quad \forall \ n \in \mathbb{I} \)
- \( 1 + \omega + \omega^2 = 0 \)
- \( \omega^3 = 1 \)
- \( \overline{\omega} = \omega^2 \) and \( \omega^2 = \omega \)
- \( \omega^2 = \frac{1}{\omega} \) and \( \frac{1}{\omega^2} = \omega \)
- \( 1 + \omega^k + \omega^{2k} = \begin{cases} 0, & \text{if } k \text{ is not multiple of 3} \\ 1, & \text{if } k \text{ is multiple of 3} \end{cases} \)
- In polar form the cube roots of unity are:

\[
\cos 0 + i \sin 0, \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}, \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}
\]

- The cube roots of unity, when represented on complex plane, lie on vertices of an equilateral triangle inscribed in a unit circle having centre at origin, one vertex being on positive real axis.

- A complex number \( a + ib \), for which \(|a : b| = 1 : \sqrt{3} \) or \( \sqrt{3} : 1 \), can always be expressed in terms of \( i, \omega, \omega^2 \).

**Note:**

If \( x, y, z \in \mathbb{R} \) and \( \omega \) is non-real cube root of unity, then

- \( x^2 + x + 1 = (x - \omega)(x - \omega^2) \)
- \( x^2 + 1 = (x + \omega)(x + \omega^2) \)
- \( x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2) \)
- \( x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2) \)
- \( x^2 + y^2 = (x + iy)(x - iy) \)
- \( x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + y\omega + z\omega^2)(x + \omega y + \omega^2 z) \)

(10) **n\(^{th}\) Roots of Unity:**

The \( n\) roots of unity are given by the solution set of the equation

\[
x^n = 1 = \cos 0 + i \sin 0 = \cos 2k\pi + i \sin 2k\pi
\]

\[
x = [\cos 2k\pi + i \sin 2k\pi]^{1/n}
\]

\[
x = \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad \text{where } k = 0, 1, 2, \ldots, (n - 1).
\]

**Properties of \( n\)\(^{th}\) roots of unity**

(i) Let \( \alpha = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} = e^{(2\pi/n)} \), then \( n\)\(^{th}\) roots of unity can be expressed in the form of a G.P. with common ratio \( \alpha \). (i.e. 1, \( \alpha \), \( \alpha^2 \), \ldots\, \( \alpha^{n-1} \).)
(ii) The sum of all \( n \) roots of unity is zero i.e., \( 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0 \)

(iii) Product of all \( n \) roots of unity is \((-1)^{n-1}\).

(iv) Sum of \( p^{th} \) power of \( n^{th} \) roots of unity

\[
1 + \alpha^p + \alpha^{2p} + \ldots + \alpha^{(n-1)p} = \begin{cases} 0, & \text{when } p \text{ is not multiple of } n \\ n, & \text{when } p \text{ is a multiple of } n \end{cases}
\]

(v) The \( n, n^{th} \) roots of unity if represented on a complex plane locate their positions at the vertices of a regular plane polygon of \( n \) sides inscribed in a unit circle having centre at origin, one vertex on vertex on positive real axis.

(11) Rotation Theorem :

(i) If \( P(z_1) \) and \( Q(z_2) \) are two complex numbers such that \( |z_1| = |z_2| \), then:

\[
\frac{z_2}{z_1} = e^{i\theta}, \text{ where } \angle POQ = \theta
\]

(ii) If \( P(z_1), Q(z_2) \) and \( R(z_3) \) are three complex numbers and \( \angle PQR = \theta \), then:

\[
\frac{z_3 - z_2}{z_1 - z_2} = \left(\frac{RQ}{PQ}\right) e^{i\theta} = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}
\]

(12) Logarithm Of A Complex Number :

Let \( z = x + iy \), then \( z = |z|(e)^{i\theta} \)

\[\therefore \log_e z = \log_e |z| + \log_e e^{i\theta}\]

\[\Rightarrow \log_e z = \frac{1}{2} \log_e (x^2 + y^2) + i \arg(z)\]

(13) Standard Loci in the Argand Plane :

- If \( z \) is a variable point in the argand plane such that \( \arg(z) = \theta \), then locus of \( z \) is a straight line (excluding origin) through the origin inclined at an angle \( \theta \) with x-axis.

- If \( z \) is a variable point and \( z_1 \) is a fixed point in the argand plane such that \( \arg(z - z_1) = \theta \), then locus of \( z \) is a straight line passing through the point representing \( z_1 \).
and inclined at an angle \( \theta \) with x-axis. Note that line point \( z_1 \) is excluded from the locus.

- If \( z \) is a variable point and \( z_1, z_2 \) are two fixed points in the argand plane, then
  
  (i) \( |z - z_1| = |z - z_2| \Rightarrow \) Locus of \( z \) is the perpendicular bisector of the line segment joining \( z_1 \) and \( z_2 \).

  (ii) \( |z - z_1| + |z - z_2| = \text{constant} \neq |z_1 - z_2| \Rightarrow \) Locus of \( z \) is an ellipse.

  (iii) \( |z - z_1| + |z - z_2| = |z_1 - z_2| \Rightarrow \) Locus of \( z \) is the line segment joining \( z_1 \) and \( z_2 \).

  (iv) \( |z - z_1| - |z - z_2| = |z_1 - z_2| \Rightarrow \) Locus of \( z \) is a straight line joining \( z_1 \) and \( z_2 \) but \( z \) does not lie between \( z_1 \) and \( z_2 \).

  (v) \( |z - z_1| - |z - z_2| = \text{constant} \neq |z_1 - z_2| \Rightarrow \) Locus of \( z \) is hyperbola.

  (vi) \( |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2 \Rightarrow \) Locus of \( z \) is a circle with \( z_1 \) and \( z_2 \) extremities of diameter.

  (vii) \( |z - z_1| = k |z - z_2|, k \neq 1 \Rightarrow \) Locus of \( z \) is a circle.

  (viii) \( \arg \left( \frac{z - z_1}{z - z_2} \right) = \alpha \Rightarrow \) Locus of \( z \) is a segment of circle.

  (ix) \( \arg \left( \frac{z - z_1}{z - z_2} \right) = \pm \pi/2 \Rightarrow \) Locus of \( z \) is a circle with \( z_1 \) and \( z_2 \) as the vertices of diameter.

  (x) \( \arg \left( \frac{z - z_1}{z - z_2} \right) = 0 \text{ or } \pi \Rightarrow \) Locus of \( z \) is a straight line passing through \( z_1 \) and \( z_2 \).

  (xi) The equation of the line joining complex numbers \( z_1 \) and \( z_2 \) is given by

\[
\frac{z - z_1}{z_2 - z_1} = \frac{z - z_1}{z_2 - z_1} \text{ or } \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0
\]

(13) Geometrical Properties:

(i) Distance formula:
If \( P(z_1) \) and \( Q(z_2) \) are two complex points on complex plane then distance between \( P \) and \( Q \) is given by:

\[ PQ = |z_1 - z_2| \]

(ii) Section formula:
If \( P(z_1) \) and \( Q(z_2) \) are two complex points on complex plane and \( R(z) \) divides the line segment \( PQ \) in ratio \( m:n \), then:

\[
z = \frac{mz_2 + nz_1}{m+n} \text{ (for internal division)}
\]

\[
z = \frac{mz_2 - nz_1}{m-n} \text{ (for external division)}
\]

Note:
If \( a, b, c \) are three real numbers such that \( az_1 + bz_2 + cz_3 = 0 \), where \( a + b + c = 0 \) and \( a, b, c \) are not all simultaneously zero, then the complex numbers \( z_1, z_2 \) and \( z_3 \) are collinear.

(iii) Triangular Properties:
If the vertices \( A, B \) and \( C \) of a triangle is represented by the complex numbers \( z_1, z_2 \) and \( z_3 \) respectively and \( a, b, c \) are the length of sides, then:

- Centroid \( (Z_G) \) of the \( \triangle ABC = \frac{z_1 + z_2 + z_3}{3} \)
• Orthocentre \( (Z_o) \) of the \( \triangle ABC = \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C} \)

• Incentre \( (Z_i) \) of the \( \triangle ABC = \frac{a z_1 + b z_2 + c z_3}{a + b + c} \)

• Circumcentre of the \( \triangle ABC = \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \)

Note:
Triangle \( ABC \) with vertices \( A(z_1) \), \( B(z_2) \) and \( C(z_3) \) is equilateral if and only if
\[
\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0
\]
\[
\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1
\]

(iv) Equation of a Straight Line:

• An equation of a straight line joining the two points \( A(z_1) \) and \( B(z_2) \) is
\[
z = tz_1 + (1 - t)z_2
\]
where \( t \) is a real parameter.

• The general equation of a straight line is \( a \bar{z} + a\bar{z} + b = 0 \) where \( a \) is a non-zero complex number and \( b \) is a real number.

(v) Complex Slope of a Line:
If \( A(z_1) \) and \( B(z_2) \) are two points in the complex plane, then complex slope of \( AB \) is defined to be \( \mu = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} \) for two lines with complex slopes \( \mu_1 \) and \( \mu_2 \) are

• parallel, if \( \mu_1 = \mu_2 \)
• perpendicular, if \( \mu_1 + \mu_2 = 0 \) the complex slope of the line \( \bar{a}z + a\bar{z} + b = 0 \) is given by \( -(a/a) \)

(vi) Length of Perpendicular from a Point to a Line:
Length of perpendicular of point \( A(\omega) \) from the line \( a\bar{z} + a\bar{z} + b = 0 \), where \( a \in C - \{0\} \), \( b \in R \), is given by : 
\[
p = \frac{|a\omega + a\bar{\omega} + b|}{2|a|}
\]

(vii) Equation of Circle:
• An equation of the circle with centre at \( z_0 \) and radius \( r \) is
\[
|z - z_0| = r
\]
or \( z = z_0 + re^{i\theta}, \ 0 \leq \theta < 2\pi \) (parametric from)
or \( zz - z_0 z_0 z + z_0 z_0 - r^3 = 0 \)
• General equation of a circle is \( zz + a\bar{z} + \bar{a}z + b = 0 \) \( (1) \)
where \( a \) is a complex number and \( b \) is a real number such that \( a\bar{a} - b \geq 0 \).
Centre of \( (1) \) is \(-a\) and its radius is \( \sqrt{a\bar{a} - b} \).
• Diameter Form of a Circle
An equation of the circle one of whose diameter is the segment joining \( A(z_1) \) and \( B(z_2) \) is
(z - z_1)(\overline{z} - \overline{z}_2) + (\overline{z} - \overline{z}_1)(z - z_2) = 0

- An equation of the circle passing through two points A(z_1) and B(z_2) is

\[
(z - z_1)(\overline{z} - \overline{z}_2) + (\overline{z} - \overline{z}_1)(z - z_2) + k \begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z}_1 & 1 \\ z_2 & \overline{z}_2 & 1 \end{vmatrix} = 0 \text{ where } k \text{ is a parameter}
\]

- Equation of a circle passing through three non-collinear points.
Let three non-collinear points be A(z_1), B(z_2) and C(z_3). Let P(z) be any point on the circle. Then either \( \angle ACB = \angle APB \) [when angles are in the same segment]
\( \angle ACB + \angle APB = \pi \) [when angles are in the opposite segment]

\[
\Rightarrow \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) - \arg\left(\frac{z - z_2}{z - z_1}\right) = 0
\]
\[
\Rightarrow \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) + \arg\left(\frac{z - z_1}{z - z_2}\right) = \pi
\]

\[
\Rightarrow \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) \left(\frac{z - z_1}{z - z_2}\right) = 0
\]
or\[
\Rightarrow \arg\left(\frac{z - z_1}{z - z_2}\right) \left(\frac{z_3 - z_2}{z_3 - z_1}\right) = \pi
\]

[using \( \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \) and \( \arg(z_1z_2) = \arg(z_1) + \arg(z_2) \)]

In any case , we get \( \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} \) is purely real.

\[
\Leftrightarrow \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(\overline{z} - \overline{z}_1)(\overline{z}_3 - \overline{z}_2)}{(\overline{z} - \overline{z}_2)(\overline{z}_3 - \overline{z}_1)}
\]

- Condition for four points to be concyclic. Four points \( z_1, z_2, z_3 \) are \( z_4 \) will lie on the same circle if and only if \( \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)} \) is purely real.

\[
\Leftrightarrow \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \text{ is purely real.}
\]
4. BINOMIAL THEOREM

An algebraic expression which contains two distinct terms is called a binomial expression.

For example: \((2x + y), \left( x + \frac{3}{y} \right), \left( \frac{3}{x} + \sqrt{x+1} \right)\) etc.

General form of binomial expression is \((a+b)n\), \(n \in \mathbb{N}\) is called the binomial theorem.

For example:

\((a + b)^1 = a + b\)
\((a + b)^2 = a^2 + 2ab + b^2\)
\((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\)

Note:
Coefficients in the binomial expansion are binomial coefficients and these coefficients have a fixed pattern which can be seen through the Pascal triangle.

\[
\begin{array}{cccccccc}
& & & & & 1 & & \\
& & & & 1 & & 1 & \\
& & & 1 & & 2 & & 1 \\
& & 1 & & 3 & & 3 & & 1 \\
& 1 & & 4 & & 6 & & 4 & & 1 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
& 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\end{array}
\]

(2) Statement of Binomial theorem:

If \(a, b \in \mathbb{C}\) and \(n \in \mathbb{N}\), then:

\[(a + b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}a^0 b^n\]

or \(\Rightarrow (a+b)^n = \sum_{r=0}^{n} \binom{n}{r}a^{n-r}b^r\)

General term in binomial expansion \((a + b)^n\) is given by \(T_{r+1}\), where \(T_{r+1} = \binom{n}{r}a^{n-r}b^r\)

Now, putting \(a = 1\) and \(b = x\) in the binomial theorem

\[(1 + x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n\]

\(\Rightarrow (1 + x)^n = \sum_{r=0}^{n} \binom{n}{r}x^r\)

General term in binomial expansion \((1 + x)^n\) is given by \(T_{r+1}\), where \(T_{r+1} = \binom{n}{r}x^r\)

(3) Properties of Binomial Theorem:

(i) The number of terms in the expansion is \(n + 1\).

(ii) The sum of the indices of \(a\) and \(b\) in each term is \(n\).

(iii) The binomial coefficients (i.e. \(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\)) of the terms equidistant from the beginning and the end are equal, i.e. \(\binom{n}{0} = \binom{n}{n}, \binom{n}{1} = \binom{n}{n-1}\) etc. \(\{\therefore \binom{n}{r} = \binom{n}{n-r}\}\)

(iv) Middle term (s) in expansion of \((a + b)^n\), \(n \in \mathbb{N}\):
If \( n \) is even, then number of terms are odd and in this case only one middle term exists which is \( \left( \frac{n+2}{2} \right)^{th} \) term. If \( n \) is odd, then number of terms are even and in this case two middle terms exist which are \( \left( \frac{n+2}{2} \right)^{th} \) and \( \left( \frac{n+3}{2} \right)^{th} \) terms.

7. **Properties of Binomial coefficients**:

\[
(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n
\]

(1) The sum of the binomial coefficients in the expansion of \( (1 + x)^n \) is \( 2^n \).

Putting \( x = 1 \) in (1)

\[\sum_{r=0}^{n} \binom{n}{r} = 2^n\]

(2) Again putting \( x = -1 \) in (1), we get

\[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} = 0\]

or \[\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0\]

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to \( 2^{n-1} \).

from (2) and (3)

\[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1}\]

\[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = 2^{n-1}\]

(4) Sum of two consecutive binomial coefficients

\[\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}\]

L.H.S. = \[\binom{n}{r} + \binom{n}{r-1} = \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!}\]

\[= \frac{n!}{(n-r)!(r-1)!} \left[ \frac{1}{r} + \frac{1}{n-r+1} \right]\]

\[= \frac{n!(n+1)}{(n-r)!(r-1)!r(r-n+1)} = \binom{n+1}{r} = R.H.S.\]

(5) Ratio of two consecutive binomial coefficients

\[\frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{n-r+1}{r}\]

(6) \[\binom{n}{r} = \frac{\binom{n-1}{r-1}}{r(r-1)} \quad \binom{n-2}{r-2} = \ldots = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{r(r-1)(r-2)\ldots21}\]
8. Binomial Theorem For Negative Integer Or Fractional Indices

If \( n \in \mathbb{R} \) then,

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots \\
\ldots \ldots \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} x^r + \ldots \ldots \infty.
\]

Remarks

(i) The above expansion is valid for any rational number other than a whole number if \(|x| < 1\).

(ii) When the index is a negative integer or a fraction the number of terms in the expansion of \((1 + x)^n\) is infinite, and the symbol \( \binom{n}{r} \) cannot be used to denote the coefficient of the general term.

(iii) The first terms must be unity in the expansion, when index 'n' is a negative integer or fraction

(iv) The general term in the expansion of \((1 + x)^n\) is \( T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} x^r \)

(v) When 'n' is any rational number other than whole number then approximate value of \((1 + x)^n\) is \(1 + nx(x^2 \text{ and higher powers of } x \text{ can be neglected})\)

(vi) Expansions to be remembered (\(|x| < 1\))

(a) \((1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots \ldots + (-1)^r x^r + \ldots \ldots \infty\)

(b) \((1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots \ldots \ldots + x^r + \ldots \ldots \infty\)

(c) \((1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \ldots + (-1)^r (r + 1) x^r + \ldots \ldots \infty\)

(d) \((1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \ldots + (r + 1)x^r + \ldots \ldots \infty\)
5. PERMUTATION AND COMBINATION

Permutations are arrangements and combinations are selections. In this chapter we discuss the methods of counting of arrangements and selections. The basic results and formulas are as follows:

(1) **Fundamental Principle of Counting**:

   (i) **Principle of Multiplication**:

   If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then total number of different ways of simultaneous occurrence of both the events in a definite order is \( m + n \).

   (ii) **Principle of Addition**:

   If an event can occur in 'm' different ways, and another event can occur in 'n' different ways, then exactly one of the events can happen in \( m + n \) ways.

(2) **The Factorial**:

Let \( n \) be a positive integer. Then the continued product of first \( n \) natural numbers is called factorial \( n \), be denoted by \( n! \) or \( n \). Also, we define \( 0! = 1 \).

When \( n \) is negative or a fraction, \( n! \) is not defined.
Thus, \( n! = n(n-1)(n-2)\ldots3.2.1 \).
Also, \( 1! = 1 \times (0!) \Rightarrow 0! = 1 \).

(3) **Arrangement or Permutation**:

The number of permutations of \( n \) different things, taking \( r \) at a time without repetition is given by \( \frac{n!}{(n-r)!} \)

\[ nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!} \]

**Note**:

(i) The number of permutations of \( n \) different things, taking \( r \) at a time with repetition is given by \( n^r \)

(ii) The number of arrangements that can be formed using \( n \) objects out of which \( p \) are identical (and of one kind) \( q \) are identical (and of another kind), \( r \) are identical (and of another kind) and the rest are distinct is \( \frac{n!}{p!q!r!} \).

(4) **Conditional Permutations**:

(i) Number of permutation of \( n \) dissimilar things taken \( r \) at a time which \( p \) particular things always occur \( = \binom{n-p}{r} r! \)

(ii) Number of permutation of \( n \) dissimilar things taken \( r \) at a time which \( p \) particular things never occur \( = \binom{n-p}{r} r! \)

(iii) The total number of permutation of \( n \) dissimilar things taken not more than \( r \) at a time,
when each thing may be repeated any number of times, is \( \frac{n(n-1)}{n-1} \).

(iv) Number of permutations of \( n \) different things, taken all at a time, when \( m \) specified things always come together is \( m! \times (n-m+1)! \).

(v) Number of permutation of \( n \) different things, taken all at a time when \( m \) specified things never come together is \( n! - m! \times (n-m+1)! \).

(5) **Circular Permutation:**

The number of circular permutations of \( n \) different things taken all at a time is given by \( (n-1)! \), provided clockwise and anti-clockwise circular permutations are considered to be different.

**Note:**
(i) The number of circular permutations of \( n \) different things taken all at a time, clockwise and anti-clockwise circular permutations are considered to be same, is given by \( \frac{(n-1)!}{2} \).

(6) **Selection or Combinations:**

The number of combinations of \( n \) different things taken \( r \) at a time is given by \( ^nC_r \), where

\[ ^nC_r = \frac{n!}{r!(n-r)!} = \frac{n^P_r}{r!} \]

where \( r \leq n \); \( n \in \mathbb{N} \) are \( r \in \mathbb{W} \).

**Note:**
(i) \( ^nC_r \) is a natural number. (ii) \( ^nC_0 = ^nC_n = 1 \), \( ^nC_r = n \)

(iii) \( ^nC_r = ^nC_{n-r} \) (iv) \( ^nC_r + ^nC_{r-1} = ^{n+1}C_r \)

(v) \( ^nC_r = ^nC_n \Leftrightarrow x = y \) or \( x + y = n \)

(vi) If \( n \) is even then the greatest value of \( ^nC_r \) is \( ^nC_{n/2} \)

(vii) If \( n \) is odd then the greatest value of \( ^nC_r \) is \( ^nC_{(n+1)/2} \) or \( ^nC_{(n-1)/2} \)

(viii) \( ^nC_r = \frac{n}{r} \times ^{n-1}C_{r-1} \)

(ix) \( ^nC_r = \frac{n-r+1}{r} \)

(x) \( ^nC_r = \frac{r+1}{n+1} \times ^{n+1}C_{r+1} \)

(xii) \( ^nC_0 + ^nC_1 + ^nC_2 + \ldots + ^nC_n = 2^n \)

(7) **Selection of one or more objects**

(a) Number of ways in which atleast one object be selected out of 'n' distinct objects is

\[ ^nC_1 + ^nC_2 + ^nC_3 + \ldots \ldots + ^nC_n = 2^n - 1 \]

(b) Number of ways in which atleast one object may be selected out of 'p' alike objects of one type 'q' alike objects of second type and 'r' alike of third type is

\[ (p+1)(q+1)(r+1)-1 \]

(c) Number of ways in which atleast one object may be selected from 'n' objects where 'p' alike of one type 'q' alike of second type and 'r' alike of third type and rest \( n-(p+q+r) \) are different, is

\[ (p+1)(q+1)(r+1) \times 2^{n-(p+q+r)-1} \]

(8) **Formation of Groups:**

Number of ways in which \( (m+n+p) \) different things can be divided into three different
groups containing m, n & p things respectively is \( \frac{(m+n+p)!}{m!n!p!} \),

If \( m = n = p \) and the groups have identical qualitative characteristic then the number of groups is \( \frac{(3n)!}{n!n!n!3!} \).

However, if \( 3n \) things are to be divided equally among three people then the number of ways is \( \frac{(3n)!}{(n!)^3} \).

8. **Multinomial Theorem**:

Coefficient of \( x^r \) in expansion of \( (1 - x)^{-n} = \binom{n+r-1}{n} (n \in \mathbb{N}) \)

Number of ways in which it is possible to make a selection from \( m + n + p = N \) things, where \( p \) are alike of one kind, \( m \) alike of second kind & \( n \) alike of third kind taken \( r \) at a time is given by coefficient of \( x^r \) in the expansion of

\[ (1 + x + x^2 + \ldots + x^{p}) (1 + x + x^2 + \ldots + x^{m}) (1 + x + x^2 + \ldots + x^{n}). \]

(i) For example the number of ways in which a selection of four letters can be made from the letters of the word \textsc{proportion} is given by coefficient of \( x^4 \) in

\[ (1 + x + x^2 + x^3) (1 + x + x^2) (1 + x + x^2) (1 + x) (1 + x). \]

(ii) **Method of fictitious partition**:

Number of ways in which \( n \) identical things may be distributed among \( p \) persons if each person may receive one, one or more things is; \( \frac{n+p-1}{n} \binom{n}{n} \)

9. Let \( N = p^a q^b r^c \ldots \) where \( p, q, r \ldots \) are distinct primes & \( a, b, c \ldots \) are natural numbers then :

(a) The total numbers of divisors of \( N \) including 1 & \( N \) is \( = (a +1)(b + 1)(c + 1) \ldots \)

(b) The sum of these divisors is = \( (p^0 + p^1 + p^2 + \ldots + p^a) (q^0 + q^1 + q^2 + \ldots + q^b) (r^0 + r^1 + r^2 + \ldots + r^c) \ldots \)

(c) Number of ways in which \( N \) can be resolved as a product of two factors is

\[ \frac{1}{2} [(a+1)(b+1)(c+1)\ldots +1] \] if \( N \) is a perfect square

\[ \frac{1}{2} [(a+1)(b+1)(c+1)\ldots +1] \] if \( N \) is a perfect square

(d) Number of ways in which a composite number \( N \) can be resolved into two factors which are relatively prime (or coprime) to each other is equal to \( 2^{n-1} \) where \( n \) is the number of different prime factors in \( N \).

10. Let there be 'n' types of objects, with each type containing at least \( r \) objects. Then the number of ways of arranging \( r \) objects in a row is \( n^r \).

11. **Dearrangement**:

Number of ways in which 'n' letters can be put in 'n' corresponding envelopes such that no letter goes to correct envelope is

\[ n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \ldots \ldots \ldots + (-1)^n \frac{1}{n!} \right). \]
Exponent of Prime $p$ in $n!$

If $p$ is a prime number and $n$ is a positive integer, then $E_p(n)$ denotes the exponent of the prime $p$ in the positive integer $n$.

$$E_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots $$

where $S$ is the largest natural number. Such that $p^S \leq n < p^{S+1}$.

Some Important Results for Geometrical problems

(i) Number of total different straight lines formed by joining the $n$ points on a plane of which $m (< n)$ are collinear is \( ^nC_2 - ^mC_2 + 1 \).

(ii) Number of total triangle formed by joining the $n$ points on a plane of which $m (< n)$ are collinear is \( ^nC_3 - ^mC_3 \).

(iii) Number of diagonals in a polygon of $n$ sides is \( ^nC_2 - n \).

(iv) If $m$ parallel lines in a plane are intersected by a family of other $n$ parallel line. Then total number of parallelograms so formed is \( ^mC_2 \times ^nC_2 \) i.e. \( \frac{mn(m-1)(n-1)}{4} \).

(v) Given $n$ points on the circumference of a circle, then

(a) Number of straight lines = \( ^nC_2 \)  
(b) Number of triangles = \( ^nC_3 \)

(c) Number of quadrilaterals = \( ^nC_4 \).

(vi) If $n$ straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. then the number of part into which these lines divide the plane is \( = 1 + \sum n \).

(vii) Number of rectangle of any size in a square of $n \times n$ is \( \sum_{r=1}^{n} r^3 \) and number of squares of any size is \( \sum_{r=1}^{n} r^2 \).

(viii) In a rectangle of $n \times p (n < p)$ number of rectangle of any size is \( \frac{np}{4} (n+1)(p+1) \) and number of squares of any size is \( \sum_{r=1}^{n} (n+1-r)(p+1-r) \).
6. PROBABILITY

Theory of probability measures the degree of certainty or uncertainty of an event, if an event is represented by 'E', then its probability is given by notation $P(E)$, where $0 \leq P(E) \leq 1$.

(1) Basic terminology:

(i) Random Experiment:
It is a process which results in an outcome which is one of the various possible outcomes that are known to happen, for example: throwing of a dice is a random experiment as it results in one of the outcome from $\{1, 2, 3, 4, 5, 6\}$, similarly taking a card from a pack of 52 cards is also a random experiment.

(ii) Sample Space:
It is the set of all possible outcomes of a random experiment for example: $\{H, T\}$ is the sample space associated with tossing of a coin. In set notation it can be interpreted as the universal set.

(iii) Event:
It is subset of sample space. for example: getting a head in tossing a coin or getting a prime number is throwing a die. In general if a sample space consists 'n' elements, then a maximum of $2^n$ events can be associated with it.

Note:
- Each element of the sample space is termed as the sample point or an event point.
- The complement of an event 'A' with respect to a sample space S is the set of all elements of 'S' which are not in A. It is usually denoted by $A'$, $\overline{A}$ or $A^c$.
- $A \cup \overline{A} = S \Rightarrow P(A) + P(\overline{A}) = 1$

(iv) Simple Event or Elementary Event:
If an event covers only one point of sample space, then it is called a simple event, for example: getting a head followed by a tail in throwing of a coin 2 times is a simple event.

(v) Compound Event or mixed event (Composite Event):
when two or more than two events occur simultaneously then event is said to be a compound event and it contains more than one element of sample space. for example: when a dice is thrown, the event of occurrence of an odd number is mixed event.

(vi) Equally likely Events:
If events have same chance of occurrence, then they are said to be equally likely events. For example:
- In a single toss of a fair coin, the events $\{H\}$ and $\{T\}$ are equally likely.
- In a single throw of an unbiased dice the events $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$ and $\{6\}$ are equally likely.
- In tossing a biased coin the events $\{H\}$ and $\{T\}$ are not equally likely.

(vii) Mutually Exclusive / Disjoint / Incompatible Events:
Two events are said to be mutually exclusive if occurrence of one of them rejects the possibility of occurrence of the other (i.e. both cannot occur simultaneously).
Mathematically, $A \cap B = \emptyset \Rightarrow P(A \cap B) = 0$

![Venn Diagram: A and B are Mutually Exclusive Events](image)

For example:
- When a coin is tossed the event of occurrence of a head and the event of occurrence of a tail are mutually exclusive events.
- When a dice is thrown sample space $S = (1, 2, 3, 4, 5, 6)$,
  
  $A = \text{the event of occurrence of a number greater than } 4 = \{5, 6\}$
  
  $B = \text{the event of occurrence of an odd number} = \{1, 3, 5\}$
  
  $C = \text{the event of occurrence of an even number} = \{2, 4, 6\}$

Now $B$ and $C$ are mutually exclusive events but $A$ and $B$ are not mutually exclusive because they can occur together (when the number 5 turns up).

**(viii) Exhaustive System of Events**:

If each outcome of an experiment is associated with at least one of the events $E_1, E_2, E_3, \ldots, E_n$, then collectively the events are said to be exhaustive. Mathematically, $E_1 \cup E_2 \cup E_3 \ldots \ldots \cup E_n = S$.

For example.

In random experiment of rolling an unbiased dice, $S = \{1, 2, 3, 4, 5, 6\}$

$E_1 = \text{the event of occurrence of a prime number} = \{2, 3, 5\}$

$E_2 = \text{the event of occurrence of an even number} = \{2, 4, 6\}$

$E_3 = \text{the event of occurrence of a number less than } 3 = \{1, 2\}$

Now, $E_1 \cup E_2 \cup E_3 = S$ and hence events $E_1, E_2$ and $E_3$ are mutually exhaustive events.

**(2) Classical Definition of Probability**:

If a random experiment results in a total of $(m + n)$ outcomes which are equally likely and mutually exclusive and exhaustive and if 'm' outcomes are favourable to an event 'E' while 'n' are unfavourable, then the probability of occurrence of the event 'E', denoted by $P(E)$, is given by:

$$P(E) = \frac{m}{m+n} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} \Rightarrow P(E) = \frac{n(E)}{n(S)}$$

**Note:**

(i) In above definition, odds in favour of 'E' are $m : n$, while odds against 'E' are $n : m$.

(ii) $P(\bar{E})$ or $P(E')$ or $P(E^c)$ denotes the probability of non-occurrence of $E$.

$$P(\bar{E}) = 1 - P(E) = \frac{n}{m+n}$$

**(3) Notation of an event in set theory**:

In dealing with problems of probability, it is important to convert the verbal description of an event into its equivalent set theoretic notation. Following table illustrates the verbal description and its corresponding notation in set theory.
### Verbal description of Event

<table>
<thead>
<tr>
<th>Both A and B occurs.</th>
<th>Notation is Set theory.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atleast one of A or B occurs</td>
<td>$A \cup B$</td>
</tr>
<tr>
<td>A occurs but not B.</td>
<td>$A \cap \overline{B}$</td>
</tr>
<tr>
<td>Neither A nor B occurs.</td>
<td>$\overline{A} \cap \overline{B}$</td>
</tr>
<tr>
<td>Exactly one of A and B occurs</td>
<td>$(A \cap \overline{B}) \cup (\overline{A} \cap B)$</td>
</tr>
<tr>
<td>All three events occur simultaneously</td>
<td>$A \cap B \cap C$</td>
</tr>
<tr>
<td>Atleast one of the events occur.</td>
<td>$A \cup B \cup C$</td>
</tr>
<tr>
<td>Only A occurs and B and C don’t occur.</td>
<td>$A \cap \overline{B} \cap \overline{C}$</td>
</tr>
<tr>
<td>Both A and B occurs but C don’t occur.</td>
<td>$A \cap B \cap \overline{C}$</td>
</tr>
<tr>
<td>Exactly two of the event A, B, C occur.</td>
<td>$(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$</td>
</tr>
<tr>
<td>Atleast two of the events occur.</td>
<td>$(A \cap B) \cup (B \cap C) \cup (A \cap C)$</td>
</tr>
<tr>
<td>exactly two of the events occur.</td>
<td>$(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$</td>
</tr>
<tr>
<td>None of the events A, B, C occur.</td>
<td>$\overline{A} \cap \overline{B} \cap \overline{C} = A \cup B \cup C$.</td>
</tr>
</tbody>
</table>

### Note:

(i) De Morgan’s Law :
If A and B are two subsets of a universal set, then

$$(A \cup B)^c = A^c \cap B^c$$ and $$(A \cap B)^c = A^c \cup B^c$$.

(ii) Distributive Law :
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(iii) Venn diagram for events A, B, C :

![Venn Diagram](image)

*Er. L.K.Sharma*
9810277682/8398015058
(4) **Addition theorem of probability**: 

If 'A' and 'B' are any two events associated with a random experiment, then

(i) \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \) or \( P(A + B) = P(A) + P(B) - P(AB) \).

(ii) \( P(A \cap B) = P(B) - P(A \cap B) \)

(iii) \( P(A \cap \overline{B}) = P(A) - P(A \cap B) \)

(iv) \( P \left( (A \cap B) \cup (\overline{A} \cap B) \right) = P(A) + P(B) - 2P(A \cap B) = P(AB) + P(A \cup B) - P(A \cap B) \).

If A , B , C are three events associated with a random experiment, then

(v) \( P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \)

(vi) \( P(\text{at least two of } A, B, C \text{ occur}) = P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C) \)

(vii) \( P(\text{exactly two of } A, B, C \text{ occur}) = P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C) \)

(viii) \( P(\text{exactly one of } A, B, C \text{ occur}) = P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C) \)

(5) **Independent and dependent events**

If two events are such that occurrence or non-occurrence of one does not effect the chances of occurrence or non-occurrence of the other event, then the events are said to be independent. Mathematically : if \( P(A \cap B) = P(A).P(B) \) then A and B are independent.

For example

- When a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.
- When two cards are drawn out of a full pack of 52 playing cards with replacement (the first card drawn is put back in the pack and the second card is drawn), then the event of occurrence of a king in the first draw and the event of occurrence of a king in the second draw are independent events because the probability of drawing a king in the second draw is \( \frac{4}{52} \) whether a king is drawn in the first draw or not. But if the two cards are drawn without replacement then the two events are not independent.

**Note:**

(i) If A and B are independent , then A' and B' are independent , A and B' are independent and A' and B are independent.

(ii) Three events A,B and C are independent if and only if all the three events are pairwise independent as well as mutually independent.

(iii) **Complementation Rule** :

If A and B are independent events, then

\[
P(A \cup B) = 1 - P(\overline{A} \cup \overline{B})
\]

\[
= 1 - P(\overline{A} \cap \overline{B})
\]

\[
= 1 - P(\overline{A}).P(\overline{B})
\]

Similarly, if A, B and C are three independent events, then

\[
P(A \cup B \cup C) = 1 - P(\overline{A}).P(\overline{B}).P(\overline{C})
\]

(6) **Conditional Probability**

Let A and B be any two events, B \( \neq \phi \), then \( P(A/B) \) denotes the conditional probability of occurrence of event A when B has already occurred.
For example:
When a die is thrown, sample space $S = \{1, 2, 3, 4, 5, 6\}$
Let $A =$ the event of occurrence of a number greater than 4.
= $\{5, 6\}$
$B =$ the event of occurrence of an odd number.
= $\{1, 3, 5\}$
Then $P(A/B) =$ probability of occurrence of a number greater than 4, when an odd number has occurred

Note:
(i) If $B \neq \emptyset$, then $P\left(\frac{A}{B}\right) + P\left(\frac{\bar{A}}{B}\right) = 1$ and $P\left(\frac{A}{B}\right) + P\left(\frac{\bar{A}}{B}\right) = 1$.

(ii) If $A \neq \emptyset$, and $B$ is dependent on $A$, then
$$P(B) = P(A)P\left(\frac{B}{A}\right) + P(\bar{A})P\left(\frac{B}{\bar{A}}\right).$$

(7) Multiplication Theorem of Probability:

If $A$ and $B$ are any two events, then
$$P(A \cap B) = P(B)P\left(\frac{A}{B}\right),$$
where $B \neq \emptyset$ and $P\left(\frac{A}{B}\right)$ denotes the probability of occurrence of event $A$ when $B$ has already occurred.
If $A$ and $B$ are independent events, then probability of occurrence of event $A$ is not affected by occurrence or non occurrence of event $B$, therefore
$$P\left(\frac{A}{B}\right) = P(A)$$

(8) Total Probability Theorem

If an event $A$ can occur with one of the $n$ mutually exclusive and exhaustive events $B_1, B_2, ..., B_n$ and the probabilities $P(A/B_1), P(A/B_2), \ldots, P(A/B_n)$ are known, then
$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i).$$

(9) Bayes' Theorem (Inverse Probability):

Let $B_1, B_2, B_3, \ldots, B_n$ is a set of $n$ mutually exclusive and exhaustive events and event $A$ can occur with any of the event $B_1, B_2, \ldots, B_n$. If event 'A' has occurred, then probability that event 'A' had occurred with event $B_i$ is given by $P\left(\frac{B_i}{A}\right)$, where
$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i)P\left(\frac{A}{B_i}\right)}{\sum_{i=1}^{n} P(B_i)P\left(\frac{A}{B_i}\right)}$$
**Particular Case:**
Let $A$, $B$, $C$ be three mutually exclusive and exhaustive events and event 'E' can occur with any of the events $A$, $B$ and $C$ as shown in the diagram:

Now, if event 'E' had occurred, then

$$P\left( \frac{A}{E} \right) = \frac{y}{x+y+z}$$

$$\Rightarrow P\left( \frac{A}{E} \right) = \frac{P(A)P\left( \frac{E}{A} \right)}{P(A)P\left( \frac{E}{A} \right) + P(B)P\left( \frac{E}{B} \right) + P(C)P\left( \frac{E}{C} \right)}$$

**10) Binomial Probability Distribution:**

Let a random experiment is conducted for $n$ trials and in each trial an event 'E' is defined for which the probability of its occurrence is 'p' and probability of non-occurrence is 'q', where $p+q = 1$. Now if in $n$ trials, event 'E' occurs for $r$ times, then the probability of occurrence of event 'E' for $r$ times is given by:

$$P(r) = ^nC_r (p)^r (q)^{n-r}.$$  

**Note:**

(i) $(q+p)^n = ^nC_0 q^n + ^nC_1 (p)(q)^{n-1} + ^nC_2 (p)^2 (q)^{n-2} + \ldots \ldots \ldots \ldots ^nC_n (p)^n$

$$\Rightarrow p(0) + p(1) + p(2) + \ldots \ldots \ldots p(n) = 1$$

(ii) For Binomial Probability distribution, its mean and variance is given by $np$ and $npq$ respectively.

**10) Geometrical Applications:**

If the number of sample points in sample space is infinite, then classical definition of probability can’t be applied. For uncountable uniform sample space, probability $P$ is given by:

$$P = \frac{\text{favourable dimension}}{\text{total dimension}}$$

For example:

- If a point is taken at random on a given straight line segment $AB$, the chance that it falls on a particular segment $PQ$ of the line segment is $PQ/AB$.

- If a point is taken at random on the area $S$ which includes an area $\Delta$, the chance that the point falls on $\Delta$ is $\Delta/S$.  

Er. L.K.Sharma
9810277682
8398015058
Any rectangular arrangement of numbers (real or complex) is called a matrix. If a matrix has 'm' rows and 'n' columns then the order of matrix is denoted by \( m \times n \).

If \( A = [a_{ij}]_{m \times n} \), where \( a_{ij} \) denotes the element of \( i^{th} \) row and \( j^{th} \) column, then it represents the following matrix:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{ij} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{1} & a_{2} & a_{3} & \cdots & a_{j} & \cdots & a_{n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}
\]

(1) Basic Definitions:

(i) Row matrix: A matrix having only one row is called a row matrix.

General form of row matrix is \( A = [a_{11} \ a_{12} \ a_{13} \ \cdots \ a_{1n}] \)

(ii) Column matrix: A matrix having only one column is called a column matrix.

General form of column matrix is \( A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \)

(iii) Singleton matrix: If in a matrix there is only one element then it is called a singleton matrix. 

\( A = [a_{ij}]_{m \times n} \) is a singleton matrix if \( m = n = 1 \). For example: \([2]\), \([3]\), \([a]\), \([-3]\) are singleton matrices.

(iv) Square matrix: A matrix in which the number of rows and columns are equal is called a square matrix. General form of a square matrix is \( A = [a_{ij}]_{n} \).

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

Note:

(i) In a square matrix, the elements of the form \( a_{ii} \) is termed as diagonal elements, where \( i = 1, 2, 3, \ldots, n \) and line joining these elements is called the principal diagonal or leading diagonal or main diagonal.
(ii) Summation of the diagonal elements of a square matrix is termed as its trace.

\[
\text{If } A = [a_{ij}]_{n \times n}, \text{ then } \text{tr}(A) = \sum_{i=1}^{n} a_{ii}
\]

\[
\Rightarrow \text{tr}(A) = a_{11} + a_{22} + a_{33} + \ldots \ldots + a_{nn}
\]

(iii) If \( A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n} \) and \( \lambda \) be a scalar, then:

- \( \text{tr}(\lambda A) = \lambda \text{tr}(A) \)
- \( \text{tr}(AB) = \text{tr}(BA) \)
- \( \text{tr}(I_n) = n \)
- \( \text{tr}(AB) \neq \text{tr}A \cdot \text{tr}B \)

(v) **Rectangular Matrix**: A matrix in which number of rows in not equal to number of columns is termed as rectangular matrix. \( A = [a_{ij}]_{m \times n} \) is rectangular if \( m \neq n \).

If \( m > n \), matrix is termed as vertical matrix and if \( m < n \), matrix is termed as horizontal matrix.

(vi) **Zero matrix**: \( A = [a_{ij}]_{m \times n} \) is called a zero matrix if \( a_{ij} = 0 \) \( \forall i \) and \( j \).

For example: \( O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \); \( O_{3 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \)

(vii) **Diagonal matrix**: A square matrix \( [a_{ij}]_{n \times n} \) is said to be a diagonal matrix if \( a_{ij} = 0 \) for \( i \neq j \). (i.e. all the elements of the square matrix other than diagonal elements are zero).

Diagonal matrix of order \( n \) is denoted as \( \text{diag} (a_{11}, a_{22}, \ldots, a_{nn}) \).

(viii) **Scalar matrix**: Scalar matrix is a diagonal matrix in which all the diagonal elements are same. \( A = [a_{ij}]_{n \times n} \) is a scalar matrix if \( a_{ij} = 0 \) for \( i \neq j \) and \( a_{ij} = k \) for \( i = j \).

(ix) **Unit matrix (Identity matrix)**: Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order \( 'n' \) is denoted by \( I_n \) (or \( I \)). \( A = [a_{ij}]_{n \times n} \) is a unit matrix if \( a_{ij} = 0 \) for \( i \neq j \) and \( a_{ii} = 1 \) for \( i = j \).

For example: \( I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), \( I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

(x) **Triangular Matrix**: A square matrix \( [a_{ij}]_{n \times n} \) is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types:

- **Upper triangular matrix**: \( A = [a_{ij}]_{n \times n} \) is said to be upper triangular if \( a_{ij} = 0 \) for \( i > j \) (i.e. all the elements below the diagonal elements are zero).

- **Lower triangular matrix**: \( A = [a_{ij}]_{n \times n} \) is said to be lower triangular matrix if \( a_{ij} = 0 \) for \( i < j \) (i.e. all the elements above the diagonal elements are zero.)
Note:

- Minimum numbers of zero in a triangular matrix is given by \( \frac{n(n-1)}{2} \), where \( n \) is order of triangular matrix.
- Diagonal matrix is both upper and lower triangular.

(xi) **Singular and Non-singular matrix**: Any square matrix \( A \) is said to be non-singular if \( |A| \neq 0 \), and a square matrix \( A \) is said to be singular if \( |A| = 0 \). \( \det(A) \) or \( |A| \) represents determinant of square matrix \( A \).

(xii) **Comparable matrices**: Two matrices \( A \) and \( B \) are said to be comparable if they have the same order.

(xiii) **Equality of matrices**: Two matrices \( A \) and \( B \) are said to be equal if they are comparable and all the corresponding elements are equal.

Let \( A = [a_{ij}]_{m \times n} \) and \( B = [b_{ij}]_{p \times q} \), then

\[ A = B \Rightarrow m = p, \ n = q \ \text{and} \ a_{ij} = b_{ij} \ \forall \ i \text{ and } j. \]

(2) **Scalar Multiplication of matrix**: 

Let \( \lambda \) be a scalar (real or complex number) and \( A = [a_{ij}]_{m \times n} \) be a matrix, then \( \lambda A \) is defined as \( [\lambda a_{ij}]_{m \times n} \).

Note:
If \( A \) and \( B \) are matrices of the same order and \( \lambda, \mu \) are any two scalars, then

(i) \( \lambda(A + B) = \lambda A + \lambda B \)

(ii) \( (\lambda + \mu)A = \lambda A + \mu A \)

(iii) \( \lambda(\mu A) = (\lambda \mu)A = \mu(\lambda A) \)

(iv) \( -\lambda A = -(\lambda A) = \lambda(-A) \)

(3) **Addition of matrices**: 

Let \( A \) and \( B \) be two matrices of same order (i.e. comparable matrices), then \( A + B \) is defined as:

\[ A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [c_{ij}]_{m \times n}, \text{ where } c_{ij} = a_{ij} + b_{ij} \ \forall \ i \text{ and } j. \]

Note:
If \( A, B \) and \( C \) are matrices of same order, then

- \( A + B = B + A \) (Commutative law)
- \( (A + B) + C = A + (B + C) \) (Associative law)
- \( A + O = O + A = A \), where \( O \) is zero matrix which is additive identity of the matrix.
- \( A + (-A) = 0 = (-A) + A \), where \( -A \) is obtained by changing the sign of every element of \( A \), which is additive inverse of the matrix.
(4) Multiplication of Matrices:

Two matrices A and B are conformable for the product AB if the number of columns in A (per-multiplier) is same as the number of rows in B (post multiplier).

Thus, if \(A = [a_{ij}]_{m \times n}\) and \(B = [b_{ij}]_{n \times p}\) are two matrices of order \(m \times n\) and \(n \times p\) respectively, then their product \(AB\) is of order \(m \times p\) and is defined as \(AB = [c_{ij}]_{m \times p}\), where \(c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj}\)

For example:
\[
\begin{pmatrix}
1 & 2 \\
-1 & 3
\end{pmatrix}_{2 \times 2} \times \begin{pmatrix}
1 & 0 & 2 \\
-2 & 1 & 1
\end{pmatrix}_{2 \times 3} = \begin{pmatrix}
1 - 4 & 0 + 2 & 2 + 2 \\
-1 - 6 & 0 + 3 & -2 + 3
\end{pmatrix}_{2 \times 3} = \begin{pmatrix}
-3 & 2 & 4 \\
-7 & 3 & 1
\end{pmatrix}_{2 \times 3}
\]

Note:
(i) If A, B and C are three matrices such that their product is defined, then
- \(AB \neq BA\) (Generally not commutative)
- \((AB)C = A(BC)\) (Associative Law)
- \(IA = A = AI\) (I is identity matrix for matrix multiplication)
- \(A(B+C) = AB + AC\) (Distributive law)
- If \(AB = AC\) \(\Rightarrow\) \(B = C\) (Cancellation law is not always applicable)
- If \(AB = 0\) (It does not imply that \(A = 0\) or \(B = 0\), product of two non-zero matrix may be a zero matrix.)

(ii) If A and B are two matrices of the same order, then
- \((A + B)^2 = A^2 + B^2 + AB + BA\)
- \((A - B)(A + B) = A^2 - B^2 + AB - BA\)
- \(A(-B) = -(AB)\)

(iii) The positive integral powers of a matrix A are defined only when A is a square matrix \(A^2 = A.A\), \(A^3 = A.A.A = A^2.A\). For any positive integers \(m, n\)

- \(A^mA^n = A^{m+n}\)
- \(I^n = I\), \(I^m = I\)
- \((A^n)^m = A^{mn} = (A^m)^n\)

\(A^n = I\) where A is a square matrix of order n.

(5) Transpose of a Matrix:

Let \(A = [a_{ij}]_{m \times n}\), then the transpose of A is denoted by \(A^t\) (or \(A^T\)) and is defined as \(A^T = [b_{ij}]_{n \times m}\), where \(b_{ij} = a_{ji}\) \(\forall\) \(i, j\).

For example: Transpose of matrix \(\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{pmatrix}_{2 \times 3}\) is \(\begin{pmatrix}
a_1 \\
b_1
\end{pmatrix}_{1 \times 2} \begin{pmatrix}
a_2 \\
b_2
\end{pmatrix}_{1 \times 2} \begin{pmatrix}
a_3 \\
b_3
\end{pmatrix}_{1 \times 2}\)

Note:
Let A and B be two matrices then
- \((A^T)^T = A\)
(6) Symmetric and Skew-symmetric Matrix:

A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji} \ \forall \ i, j$ (i.e. $A^T = A$)

For example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if $a_{ij} = -a_{ji} \ \forall \ i, j$ (i.e. $A^T = -A$)

For example: $B = \begin{bmatrix} 0 & x & y \\ -x & 0 & -z \\ -y & z & 0 \end{bmatrix}$ is a skew symmetric matrix.

Note:
- Every unit matrix, scalar matrix and square zero matrix are symmetric matrices.
- Maximum number of different elements in a symmetric matrix of order 'n' is $\frac{n(n+1)}{2}$.
- All principal diagonal elements of a skew-symmetric matrix are always zero because for any diagonal element. $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$
- Trace of a skew symmetric matrix is always 0.
- If $A$ is skew symmetric of odd order, then $\det(A) = 0$.

(7) Properties of Symmetric and skew-symmetric Matrices:

(i) If $A$ is a square matrix, then $A + A^T$, $AA^T$, $A^T A$ are symmetric matrices, while $A - A^T$ is skew-symmetric matrix.

(ii) If $A$ is a symmetric matrix, then $-A$, $KA$, $A^T$, $A^n$, $A^{-1}$, $B^T AB$ are also symmetric matrices, where $n \in \mathbb{N}$, $K \in \mathbb{R}$ and $B$ is a square matrix of order as that of $A$.

(iii) If $A$ is a skew-symmetric matrix, then
- $A^{2n}$ is a symmetric matrix for $n \in \mathbb{N}$
- $A^{2n+1}$ is a skew-symmetric matrix for $n \in \mathbb{N}$
- $kA$ is skew-symmetric matrix, where $K \in \mathbb{R}$
- $B^T AB$ is skew-symmetric matrix, where $B$ is a square matrix of order as that of $A$.

(iv) If $A$ and $B$ are two symmetric matrices, then
- $A \pm B$, $AB + BA$ are also symmetric matrices.
• AB – BA is a skew-symmetric matrix
• AB is symmetric matrix when AB = BA.

(v) If A and B are two skew-symmetric matrices, then
• A ± B, AB – BA are skew-symmetric matrices
• AB + BA is a symmetric matrix.

(vi) If A is a skew-symmetric matrix and C is a column matrix, then C^T AC is a zero matrix.

(vii) Every square matrix A can uniquely be expressed as sum of symmetric and skew-symmetric matrix (i.e. \( A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \)).

(8) **Submatrix, Minors, Cofactors:**

**Submatrix:** In a given matrix A, the matrix obtained by deleting some rows or columns (or both) of A is called as submatrix of A.

For example: If \( A = \begin{bmatrix} a & b & c & d \\ p & q & r & s \\ x & y & z & w \end{bmatrix} \), then

\[
\begin{bmatrix} a & b \\
p & q \\
x & y \end{bmatrix}, \begin{bmatrix} a & b & c \\
p & q & r \\
x & y & z \end{bmatrix}, \begin{bmatrix} a & b \\
p & q \\
x & y \end{bmatrix}, \begin{bmatrix} b & c \\
q & r \\
y & z \end{bmatrix}, \begin{bmatrix} c \\
r \\
z \end{bmatrix}, \begin{bmatrix} d \\
s \\
w \end{bmatrix}
\]

are all submatrices of A.

**Minors and Cofactors:** If \( A = [a_{ij}] \) is a square matrix, then minor of element \( a_{ij} \), denoted by \( M_{ij} \), is defined as the determinant of the submatrix obtained by deleting \( i^{th} \) row and \( j^{th} \) column of A.

Cofactor of element \( a_{ij} \), denoted by \( C_{ij} \), is defined as \( C_{ij} = (-1)^{i+j}M_{ij} \)

For example: If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then

\[ M_{11} = d = C_{11}, \quad M_{12} = c, \quad M_{21} = b, \quad M_{22} = a = C_{22} \]

If \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \), then

\[ M_{11} = ei - hf, \quad M_{22} = af - dc \]

and \( C_{22} = ai - cg, \quad C_{31} = bf - ec, \quad C_{23} = bg - ah \), etc.
Determinant of $A$ matrix:

If $A = [a_{ij}]$, is a square matrix of order $n > 1$, then determinant of $A$ is defined as the summation of products of elements of any one row (or any one column) with the corresponding cofactors.

For example:

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$

$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$, or

$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}.$

$= -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Properties of determinant:

- If $A = [a_{ij}]$, then the summation of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly, the summation of the products of elements of any column with corresponding cofactors of any other column is zero).

For example: $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, and $a_{12}C_{11} + a_{22}C_{21} + a_{32}C_{31} = 0$.

- $|A| = |A^T|$ for any square matrix $A$.

- $|AB| = |A||B| = |BA|$

- If $\lambda$ be a scalar, then $\lambda |A|$ is obtained by multiplying any one row (or any one column) of $|A|$ by $\lambda$.

$| \lambda A | = \lambda^n | A |$, where $A = [a_{ij}]_n$

- If $A$ is a skew symmetric matrix of odd order then $|A| = 0$

- If $A = \text{diag} (a_1, a_2, \ldots, a_n)$ then $|A| = a_1a_2\ldots a_n$

- $|A| = |A|n|A|, \ n \in \mathbb{N}$ and $|A^{-1}| = \frac{1}{|A|}$

- If two rows are identical (or two columns are identical) then $|A| = 0$

- If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value but is changed in sign only.
If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of two determinants.

The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

(10) Cofactor matrix and adjoint matrix:

If \( A = [a_{ij}] \) is a square matrix, then matrix obtained by replacing each element of \( A \) by corresponding cofactor is called as cofactor matrix of \( A \), denoted as cofactor \( A \).

The transpose of cofactor matrix of \( A \) is called as adjoint of \( A \), denoted by \( \text{adj} \ A \).

If \( A = [a_{ij}]_n \), then cofactor \( A = [C_{ij}]_n \) where \( C_{ij} \) is the cofactor of \( a_{ij} \) \( \forall \) \( i \) and \( j \).

\[ \text{adj}(A) = [C_{ji}]_n \]

If \( A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\
             a_{21} & a_{22} & a_{23} \\
             a_{31} & a_{32} & a_{33} \end{bmatrix} \), then \( \text{adj} \ A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\
                              C_{21} & C_{22} & C_{23} \\
                              C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\
                              C_{12} & C_{22} & C_{32} \\
                              C_{13} & C_{23} & C_{33} \end{bmatrix} ; \]

Where \( C_{ij} \) denotes the cofactor of \( a_{ij} \) in \( A \).

Note:

If \( A \) and \( B \) are square matrices of order \( n \) and \( I_n \) is unit matrix, then

- \( A(\text{adj} A) = |A| I_n = (\text{adj} A)A \)
- \( |\text{adj} A| = |A|^{n-1} \)
- \( |\text{adj}(\text{adj} A)| = |A|^{(n-1)^2} \)
- \( \text{adj}(AB) = (\text{adj} B)(\text{adj} A) \)
- \( \text{adj}(kA) = k^{n-1}(\text{adj} A) , \ k \in R \)
- \( \text{adj}(O) = O \)
- \( A \) is diagonal \( \Rightarrow \) \( \text{adj} A \) is also diagonal
- \( A \) is singular \( \Rightarrow \) \( |\text{adj} A| = 0 \)
- \( A \) is symmetric \( \Rightarrow \) \( \text{adj} A \) is also symmetric.
- \( A \) is triangular \( \Rightarrow \) \( \text{adj} A \) is also triangular.

(11) Inverse of a matrix (reciprocal matrix):

A non-singular square matrix of order \( n \) is invertible if there exists a square matrix \( B \) of the same order such that \( AB = I_n = BA \).

In such a case, we say that the inverse of \( A \) is \( B \) and we write \( A^{-1} = B \)

\[ \therefore \quad AA^{-1} = A^{-1}A = I. \]

The inverse of \( A \) is given by \( A^{-1} = \frac{1}{|A|} \text{adj} A \)

The necessary and sufficient condition for the existence of the inverse of a square matrix \( A \) is that \( |A| \neq 0 \)

Note:

If \( A \) and \( B \) are invertible matrices of the same order, then
• \((A^{-1})^{-1} = A\)
• \((A^T)^{-1} = (A^{-1})^T\)
• \((AB)^{-1} = B^{-1}A^{-1}\)
• \((A^k)^{-1} = (A^{-1})^k, k \in \mathbb{N}\)

\[\text{adj}(A^{-1}) = (\text{adj} A)^{-1}\]

\[|A^{-1}| = \frac{1}{|A|} = |A|^{-1}\]

• \(A = \text{diag}(a_1, a_2, ..., a_n) \implies A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, ..., a_n^{-1})\)

• \(A\) is symmetric \(\implies A^{-1}\) is also symmetric.
• \(A\) is diagonal, \(|A| \neq 0\) \(\implies A^{-1}\) is also diagonal.
• \(A\) is scalar matrix \(\implies A^{-1}\) is also scalar matrix.
• \(A\) is triangular, \(|A| \neq 0\) \(\implies A^{-1}\) is also triangular.

• Every invertible matrix possesses a unique inverse.

• If \(A\) is a non-singular matrix, then \(AB = AC \implies B = C\) and \(BA = CA \implies B = C\).

• In general \(AB = O\) does not imply \(A = O\) or \(B = O\). But if \(A\) is non singular and \(AB = O\), then \(B = O\). Similarly \(B\) is non singular and \(AB = O \implies A = O\). Therefore, \(AB = O \implies \) either both are singular or one of them is \(O\).

12 Important Definitions on Matrices:

(i) Orthogonal matrix:
A square matrix \(A\) is said to be an orthogonal matrix if, \(AA^T = I = A^T A\).

For example: \[
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

(ii) Involutory matrix:
A square matrix \(A\) is said to be involutory if \(A^2 = I\), \(I\) being the identity matrix.

For example: \[
A = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
is an involutory matrix.

(iii) Idempotent matrix:
A square matrix \(A\) is said to be idempotent if \(A^2 = A\).

For example: \[
\begin{bmatrix}
1/2 & 1/2 \\
1/2 & 1/2
\end{bmatrix}
\]
is an idempotent matrix.

(iv) Nilpotent matrix:
A square matrix is said to be nilpotent of index \(p\), if \(p\) is the least positive integer such that \(A^p = O\).

For example: \[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
is nilpotent matrix of index 2.
(v) Periodic matrix:
A matrix A will be called a periodic matrix if $A^{k+1} = A$, where $k$ is a positive integer. If however $k$ is the least positive integer for which $A^{k+1} = A$, then $k$ is said to be the period of $A$.

(vi) Differentiation of a matrix:
If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$, then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is the differentiation of matrix $A$.

For example: If $A = \begin{bmatrix} x^2 & \sin x \\ 2x & 2 \end{bmatrix}$, then $\frac{dA}{dx} = \begin{bmatrix} 2x & \cos x \\ 2 & 0 \end{bmatrix}$

(vii) Conjugate of a matrix:
If matrix $A = [a_{ij}]_{m \times n}$, then the conjugate of matrix $A$ is given by $\bar{A}$, where $\bar{A} = [\bar{a}_{ij}]_{m \times n}$

For example: If $A = \begin{bmatrix} 2 + 3i & i \\ 4 & 3 - i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 2 - 3i & -i \\ 4 & 3 + i \end{bmatrix}$

(viii) Hermitian and skew-Hermitian matrix:
A square matrix $A = [a_{ij}]$ is said to be Hermitian matrix if $a_{ij} = \bar{a}_{ji} \quad \forall \quad i, j \Rightarrow A = A^\dagger$

For example: $\begin{bmatrix} a & b + ic \\ b - ic & d \end{bmatrix}$

A square matrix, $A = [a_{ij}]$ is said to be a skew-Hermitian if $a_{ij} = -\bar{a}_{ji} \quad \forall \quad i, j \Rightarrow A = -A^\dagger$

For example: $\begin{bmatrix} 0 & -2 + i \\ 2 - i & 0 \end{bmatrix}$

(ix) Unitary matrix:
A square matrix $A$ is said to be unitary if $AA^\dagger = I$, where $A^\dagger$ is the transpose of complex conjugate of $A$.

(13) System of Linear Equations and Matrices

Consider the system
\[
\begin{align*}
\ a_{11}x_1 + a_{12}x_2 + \ldots \ldots + a_{1n}x_n &= b_1 \\
\ a_{21}x_1 + a_{22}x_2 + \ldots \ldots + a_{2n}x_n &= b_2 \\
\ & \text{........................................} \\
\ a_{m1}x_1 + a_{m2}x_2 + \ldots \ldots + a_{mn}x_n &= b_n
\end{align*}
\]
Let \( A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \), \( X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) and \( B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \)

Then the above system can be expressed in the matrix form as \( AX = B \).

The system is said to be consistent if it has at least one solution.

(i) **System of linear equations and matrix inverse:**

If the above system consist of \( n \) equations in \( n \) unknowns, then we have \( AX = B \) where \( A \) is a square matrix. If \( A \) is non-singular, solution is given by \( X = A^{-1}B \).

If \( A \) is singular, \( (\text{adj} \ A) \ B = 0 \) and all the columns of \( A \) are not proportional, then the system has infinitely many solutions.

If \( A \) is singular and \( (\text{adj} \ A) \ B \neq 0 \), then the system has no solution (we say it is inconsistent).

(ii) **Homogeneous system and matrix inverse:**

If the above system is homogeneous, \( n \) equations in \( n \) unknowns, then in the matrix for \( m \) it is \( AX = 0 \). (\( \because \) in this case \( b_1 = b_2 = \ldots = b_n = 0 \) ), where \( A \) is a square matrix.

If \( A \) is non-singular, the system has only the trivial solution (zero solution) \( X = 0 \).

If \( A \) is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non trivial solutions.

(iii) **Rank of a matrix:**

Let \( A = [a_{ij}]_{m \times n} \). A natural number \( \rho \) is said to be the rank of \( A \) if \( A \) has a non-singular submatrix of order \( \rho \) and it has no nonsingular submatrix of order more than \( \rho \). Rank of zero matrix is regarded to be zero.

\[
\begin{bmatrix}
3 & -1 & 2 & 5 \\
0 & 0 & 2 & 0 \\
0 & 0 & 5 & 0
\end{bmatrix}
\]

we have \( \begin{bmatrix}
3 & 2 \\
0 & 2
\end{bmatrix} \) as a non singular submatrix.

The square matrices of order 3 are

\[
\begin{bmatrix}
3 & -1 & 2 \\
0 & 0 & 2 \\
0 & 0 & 5
\end{bmatrix}, \begin{bmatrix}
3 & -1 & 5 \\
0 & 0 & 0 \\
0 & 0 & 5
\end{bmatrix}, \begin{bmatrix}
3 & 2 & 5 \\
0 & 2 & 0 \\
0 & 5 & 0
\end{bmatrix}, \begin{bmatrix}
-1 & 2 & 5 \\
0 & 2 & 0 \\
0 & 5 & 0
\end{bmatrix}
\]

and all these are singular. Hence rank of \( A \) is 2.

(iv) **Elementary row transformation of matrix:**

The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.

(b) Multiplications of all the elements of row by a nonzero scalar.

(b) Addition of constant multiple of a row to another row.
Note: Similar to above we have elementary column transformations also.

Remark:

1. Elementary transformation on a matrix does not affect its rank.

2. Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write $A \sim B$.

(v) **Echelon form of a matrix**: A matrix is said to be in Echelon form if it satisfy the following:

(a) The first non-zero element in each row is 1 & all the other elements in the corresponding column (i.e. the column where 1 appears) are zeroes.

(b) The number of zeroes before the first non-zero element in any non-zero row is less than the number of such zeroes in succeeding non-zero rows.

Result: Rank of a matrix in Echelon form is the number of non-zero rows (i.e. number of rows with at least one non-zero element).

Remark:

1. To find the rank of a given matrix we may reduce it to Echelon form using elementary row transformations and then count the number of non-zero rows.

(vi) **System of linear equations & rank of matrix**: Let the system be $AX = B$ where $A$ is an $m \times n$ matrix, $X$ is the $n$-column vector & $B$ is the $m$-column vector. Let $[AB]$ denote the augmented matrix (i.e. matrix obtained by accepting elements of $B$ as $(n + 1)^{th}$ column & first $n$ columns are that of $A$).

$\rho(A)$ denote rank of $A$ and $\rho([AB])$ denote rank of the augmented matrix.

Clearly $\rho(A) \leq \rho([AB])$

(a) If $\rho(A) < \rho([AB])$ then the system has no solution (i.e. system is inconsistent).

(b) If $\rho(A) = \rho([AB]) = \text{number of unknowns}$, then the system has unique solution. (and hence is consistent)

(c) If $\rho(A) = \rho([AB]) < \text{number of unknowns}$, then the systems has infinitely many solutions (and so is consistent).

(vii) **Homogeneous system & rank of matrix**: Let the homogenous system be $AX = 0$, $m$ equations in $n$ unknowns. In this case $B = 0$ and so $\rho(A) = \rho([AB])$.

Hence if $\rho(A) = n$, then the system has only the trivial solution. If $\rho(A) < n$, then the system has infinitely many solutions.
Cayley-Hemilton Theorem:

Every matrix satisfies its characteristic equation e.g. let A be a square matrix then $|A-xI|=0$ is characteristics equation of A. If $x^3 - 4x^2 - 5x - 7 = 0$ is the characteristic equation for A, then $A^3 - 4A^2 + 5A - 7I = 0$

Roots of characteristic equation for A are called Eigen values of A or characteristic roots of A or latent roots of A. If $\lambda$ is characteristic root of A, then $\lambda^{-1}$ is characteristic root of $A^{-1}$. 
Consider the equations \( a_1x + b_1y = 0 \) and \( a_2x + b_2y = 0 \)

\[\Rightarrow \frac{y}{x} = -\frac{a_1}{b_1} \quad \text{and} \quad \frac{y}{x} = -\frac{a_2}{b_2}\]

\[\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}\]

\[\therefore a_1b_2 - a_2b_1 = 0\]

Now the obtained eliminant \( a_1b_2 - a_2b_1 \) is expressed as \( \begin{vmatrix} a_1 & b_1 \\ b_2 & b_2 \end{vmatrix} \), which is called the determinant of order two.

\[\therefore \begin{vmatrix} a_1 & b_1 \\ b_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1\]

(1) **Minors and Cofactors :**

If \( A = [a_{ij}] \) is a square matrix , then minor of element \( a_{ij} \), denoted by \( M_{ij} \), is defined as the determinant of the submatrix obtained by deleting \( i^{th} \) row and \( j^{th} \) column of \( A \).

Cofactor of element \( a_{ij} \), denoted by \( C_{ij} \), is defined as \( C_{ij} = (-1)^{i+j}M_{ij} \)

For example: If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then

\[M_{11} = d = C_{11}\]
\[M_{12} = c, \quad C_{12} = -c\]
\[M_{21} = b, \quad C_{21} = -b\]
\[M_{22} = a = C_{22}\]

If \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \), then

\[M_{11} = ei-hf, \quad M_{32} = af-dc\]
and \( C_{22} = ai-cg, \quad C_{31} = bf-ec, \quad C_{33} = bg-ah \), etc.

(2) **Determinant of A matrix :**

If \( A = [a_{ij}] \) be a square matrix of order \( n > 1 \) , then determinant of \( A \) is defined as the summation of products of elements of any one row (or any one column) with the corresponding cofactors.

For example: \( A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \)

\[|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \]

\[= a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32}, \text{ or}\]

\[= a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} \]
\[ |A| = a_{12} C_{12} + a_{22} C_{22} + a_{32} C_{32} \]
\[ = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \]

**Note:**

If \( A = [a_{ij}]_n \), then the summation of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly, the summation of the products of elements of any column with corresponding cofactors of any other column is zero).

For example:
\[ a_{11} C_{21} + a_{12} C_{22} + a_{13} C_{23} = 0 \]
and
\[ a_{12} C_{11} + a_{22} C_{21} + a_{32} C_{31} = 0. \]

(3) **Properties of Determinants:**

(i) The value of a determinant remains unaltered, if the rows and columns are interchanged.

\[ \text{i.e. } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_1 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \Rightarrow |A| = |A^T| \]

(ii) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. For example:

If \( D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \), \( R_1 \leftrightarrow R_2 \Rightarrow \)
\[ D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \]

then \( D' = -D \)

(iii) If a determinant has all the elements zero in any row or column then its value is zero. For example:
\[ D = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \]

(iv) If a determinant has any two rows (or columns) identical, then its value is zero.

For example: \( D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \) \( (R_1 = R_2) \)

(v) If all the elements of any row (or column) is multiplied by the same number, then the determinant is multiplied by that number. For example:

\[ D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ then } D' = KD \]

**Note:**

If \( \lambda \) be a scalar, then \( \lambda |A| \) is obtained by multiplying any one row (or any one column) of \( |A| \) by \( \lambda \).

\[ |\lambda A| = \lambda^n |A|, \text{ where } A = [a_{ij}]_n \]
(vi) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants.

For example:

\[
\begin{vmatrix}
  a_1 + x & b_1 + x & c_1 + x \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix} = \begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix} + \begin{vmatrix}
  x & y & z \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix}
\]

(vii) The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column). For example:

\[
\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix} + \begin{vmatrix}
  a_2 & b_2 & c_2 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix} = \begin{vmatrix}
  a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\
  a_2 & b_2 & c_2 \\
  a_3 + na_1 & b_3 + nb_1 & c_3 + nc_1 \\
\end{vmatrix}, \text{ then } D' = D
\]

(4) **Summation of Determinant:**

Let \( \Delta(r) = \begin{vmatrix}
  f(r) & g(r) & h(r) \\
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
\end{vmatrix} \)

where \( a_1, a_2, a_3, b_1, b_2, b_3 \) are constants (independent of \( r \)) , then

\[
\sum_{r=1}^{n} \Delta(r) = \begin{vmatrix}
  \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
\end{vmatrix}
\]

**Note:**

If more than one row or one column are function of \( r \), then first the determinant is simplified and then summation is calculated.

(5) **Integration of a Determinant:**

Let \( \Delta(x) = \begin{vmatrix}
  f(x) & g(x) & h(x) \\
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
\end{vmatrix} \)

where \( a_1, a_2, a_3, b_1, b_2, b_3 \) are constants (independent of \( x \)) , then

\[
\int_{a}^{b} \Delta(x)dx = \begin{vmatrix}
  f(x)dx & g(x)dx & h(x)dx \\
  a & a & a \\
  b & b & b \\
\end{vmatrix}
\]

**Note:**

If more than one row or one column are function of \( x \), then first the determinant is simplified and then it is integrated.
(6) **Differentiation of Determinant** :

Let \( \Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} \), then

\[
\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2'(x) & h_3'(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3'(x) \\ g_1(x) & g_2'(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3'(x) \end{vmatrix}.
\]

(7) **Multiplication Of Two Determinants** :

\[
\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 & a_1m_1 + b_1m_2 \\ a_2\ell_1 + b_2\ell_2 & a_2m_1 + b_2m_2 \end{vmatrix}.
\]

Multiplication expressed in the calculation is row \( \times \) column which also applicable in matrix multiplication, but in case of multiplication of determinants, row \( \times \) column, row \( \times \) row, column \( \times \) row and column \( \times \) column all are applicable.

**Note:**

Power cofactor formula:

If \( \Delta = |a_{ij}| \) is a determinant of order \( n \), then the value of the determinant \( |C_{ij}| = \Delta^{n-1} \), where \( C_{ij} \) is cofactor of \( a_{ij} \).

(8) **Standard Results** :

- \[
\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & c & 0 \end{vmatrix} = abc.
\]

- \[
\begin{vmatrix} a & h & g \\ h & b & f \\ g & c & 0 \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.
\]

- \[
\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc) = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)
\]

\[
= -\frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2)
\]

- \[
\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)
\]

- \[
\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)
\]

- \[
\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(ab + bc + ca)
\]
(8) **Cramer’s Rule : System of Linear equations**

(I) Consider the system of non-homogenous equations:

\[\begin{align*}
& a_1x + b_1y + c_1z = d_1 \\
& a_2x + b_2y + c_2z = d_2 \\
& a_3x + b_3y + c_3z = d_3
\end{align*}\]

Where

\[D = \begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix}, \quad D_1 = \begin{vmatrix}
  d_1 & b_1 & c_1 \\
  d_2 & b_2 & c_2 \\
  d_3 & b_3 & c_3 \\
\end{vmatrix}, \quad D_2 = \begin{vmatrix}
  a_1 & d_1 & c_1 \\
  a_2 & d_2 & c_2 \\
  a_3 & d_3 & c_3 \\
\end{vmatrix}, \quad D_3 = \begin{vmatrix}
  a_1 & b_1 & d_1 \\
  a_2 & b_2 & d_2 \\
  a_3 & b_3 & d_3 \\
\end{vmatrix}\]

according to Cramer’s rule:

\[\begin{align*}
& x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}
\end{align*}\]

Non-homogenous system of equations may or may not be consistent, following cases explains the consistency of equations:

- If \(D \neq 0\), then the system of equations are consistent and have unique solution.
- If \(D = D_1 = D_2 = D_3 = 0\), then the system of equations have either infinite solutions or no solution.
- If \(D = 0\) and atleast one of \(D_1, D_2, D_3\) is not zero then the equations are inconsistent and have no solution.

(II) Consider the system of homogenous equations:

\[\begin{align*}
& a_1x + b_1y + c_1z = 0 \\
& a_2x + b_2y + c_2z = 0 \\
& a_3x + b_3y + c_3z = 0
\end{align*}\]

In homogenous system, \(D_1 = D_2 = D_3 = 0\), hence:

- If \(D \neq 0\), then the system of equations are consistent and have unique solution (i.e. *Trivial solution*).
- If \(D = 0\), then the system of equations consistent and have infinite solutions. (i.e. *Non-trivial solution*).  

**Note:**

Homogenous system of equations are always consistent, either having Trivial solution or non-trivial solution.

(III) System of equations, \(a_1x + b_1y = c_1\) and \(a_2x + b_2y = c_2\) represents two straight lines in 2-dimensional plane, hence:

- If \(\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\): System of equations are consistent and have unique solution (i.e. Intersecting lines)
- If \(\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}\): Wywtem of equation are consistent and have infinite solution (i.e. coincident lines)
- If \(\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}\): System of equation in inconsistent and have no solution (i.e. parallel lines).

**Note:**

**Three equation in two variables:**

If \(x\) and \(y\) are not zero, then condition for \(a_1x + b_1y + c_1 = 0; a_2x + b_2y + c_2 = 0\) and
a_3x + b_3y + c_3 = 0 to be consistent in x and y is

\[
\begin{vmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    a_3 & b_3 & c_3
\end{vmatrix} = 0.
\]

(10) Application of Determinants:

(i) Area of a triangle whose vertices are \((x_1, y_1)\), \((x_2, y_2)\) and \((x_3, y_3)\) is:

\[
\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}
\]

If \(\Delta = 0\), then the three points are collinear.

(ii) Equation of a straight line passing through \((x_1, y_1)\) and \((x_2, y_2)\) is:

\[
\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \quad \text{(two point form of line)}
\]

(iii) If lines \(a_1x + b_1y + c_1 = 0\), \(a_2x + b_2y + c_2 = 0\) and \(a_3x + b_3y + c_3 = 0\) are non-parallel and non-coincident, then lines are concurrent, if

\[
\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0
\]

(iv) \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\) represents a pair of straight lines if:

\[
\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0
\]
9. SEQUENCE AND SERIES

(1) Basic Definitions:

Sequence:

A sequence is a function whose domain is the set of natural numbers. Since the domain for every sequence is the set \( \mathbb{N} \) of natural numbers, therefore a sequence is represented by its range. If \( f : \mathbb{N} \rightarrow \mathbb{R} \), then \( f(n) = T_n, \ n \in \mathbb{N} \) is called a sequence and is denoted by \( \{T_n\} \). 

\[ \{f(1), f(2), f(3), \ldots\} = \{T_1, T_2, T_3, \ldots\} \]

Note:

(i) A sequence whose range is a subset of \( \mathbb{R} \) is called a real sequence.
(ii) Finite sequences: A sequence is said to be finite if it has finite number of terms.
(iii) Infinite sequences: A sequence is said to be infinite if it has infinite number of terms.
(iv) It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the \( n \)th term. Sequences whose terms follow certain patterns are called progressions.
(v) All progressions are sequences, but all sequences are not progressions. For example: 
Set of prime numbers is a sequence but not a progression.

Series:

By adding or subtracting the terms of a sequence, we get an expression which is called a series. If \( a_1, a_2, a_3, \ldots, a_n \) is a sequence, then the expression \( a_1 + a_2 + a_3 + \ldots + a_n \) is a series.

For example:
(i) \( 1 + 2 + 3 + 4 + \ldots + n \)
(ii) \( 2 + 4 + 8 + 16 + \ldots \)

\( n^\text{th} \) term of sequence:

If summation of \( n \) terms of a sequence is given by \( S_n \), then its \( n^\text{th} \) term is given by:

\[ T_n = S_n - S_{n-1} \]

Series:

(2) Arithmetic progression (A.P.):

A.P. is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If \( a \) is the first term and \( d \) is the common difference, then A.P. can be written as:

\[ a, a + d, a + 2d, \ldots, a + (n-1)d, \ldots \]

\( n^\text{th} \) term of an A.P.:

Let \( a \) be the first term and \( d \) be the common difference of an A.P., then

\[ T_n = a + (n-1)d \text{ and } T_n - T_{n-1} \text{ = Constant. (i.e. , common difference)} \]

Sum of first \( n \) terms of an A.P.:

If \( a \) is first term and \( d \) is common difference then

\[ S_n = \frac{n}{2}(2a + (n-1)d) \Rightarrow S_n = \frac{n}{2}(a + l) \]

where \( l \) is the last term

\( p^\text{th} \) term of an A.P. from the end:

Let 'a' be the first term and 'd' be the common difference of an A.P. having \( n \) terms. Then \( p^\text{th} \) term from the end is \( (n - p + 1)^{\text{th}} \) term from the beginning.

\[ p^\text{th} \text{ term from the end} = T_{(n-p+1)} = a + (n - p)d \]
(3) Properties of A.P. :

(i) If \( a_1, a_2, a_3, \ldots \) are in A.P. whose common difference is \( d \), then for fixed non-zero number \( K \in \mathbb{R} \):
- \( a_1 \pm K, a_2 \pm K, a_3 \pm K, \ldots \) will be in A.P., whose common difference will be \( d \).
- \( Ka_1, Ka_2, Ka_3 \ldots \) will be in A.P. with common difference = \( Kd \).
- \( \frac{a_1}{K}, \frac{a_2}{K}, \frac{a_3}{K}, \ldots \) will be in A.P. with common difference = \( d/K \).
- \( \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots \) will be in H.P.
- If \( x \in \mathbb{R}^+ - \{1\} \), then \( x^{a_1}, x^{a_2}, x^{a_3}, \ldots \) will be in G.P.

(ii) The sum of terms of an A.P. which are equidistant from the beginning and the end is constant and is equal to sum of first and last term. i.e. \( a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots \)

(iii) Any term (except the first term) of an A.P. is equal to half of the sum of terms equidistant from the term i.e. \( a_n = \frac{1}{2}(a_{n-k} + a_{n+k}), k < n \).

(iv) If number of terms of any A.P. is odd, then sum of the terms is equal to product of middle term and number of terms.

(v) If number of terms of any A.P. is even then A.M. of middle two terms is A.M. of first and last term.

(vi) If the number of terms of an A.P. is odd then its middle term is A.M. of first and last term.

(vii) If \( a_1, a_2, \ldots \), \( a_n \) and \( b_1, \ldots, b_n \) are the two A.P.’s. Then \( a_1 \pm b_1, a_2 \pm b_2, \ldots, a_n \pm b_n \) are also A.P.’s with common difference \( d_1 \neq d_2 \), where \( d_1 \) and \( d_2 \) are the common difference of the given A.P.’s.

(viii) Three numbers \( a, b, c \) are in A.P. iff \( 2b = a + c \).

(ix) If the terms of an A.P. are chosen at regular intervals, then they form an A.P.

(x) Three numbers in A.P. can be taken as \( a-d, a, a+d \); four numbers in A.P. can be taken as \( a-3d, a-d, a+d, a+3d \); five numbers in A.P. are \( a-2d, a-d, a, a+d, a+2d \) and six terms in A.P. are \( a-5d, a-3d, a-d, a+d, a+3d, a+5d \) etc.

(4) Arithmetic Mean (A.M.) :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if \( a, b, c \) are in A.P., \( b \) is A.M. of \( a \) and \( c \).

\[
b = \frac{a + c}{2}
\]

n-Arithmetic Means Between Two Numbers :

If \( a, b \) are any two given numbers and \( a, A_1, A_2, \ldots, A_n, b \) are in A.P. then \( A_1, A_2, \ldots, A_n \) are the \( n \) A.M.'s between \( a \) and \( b \).

\[
A_1 = a + \frac{b-a}{n+1}, \quad A_2 = a + \frac{2(b-a)}{n+1}, \ldots, \quad A_n = a + \frac{n(b-a)}{n+1}
\]

Note:
Sum of \( n \) A.M.'s inserted between \( a \) and \( b \) is equal to \( n \) times the single A.M. between
a and b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a and b.

(5) Geometric Progression (G.P.):

G.P. is a sequence of numbers whose first term is non zero and each of the succeeding terms is equal to the preceding terms multiplied by a constant. Thus in a G.P. the ratio of $n^{th}$ term and $(n-1)^{th}$ term is constant. This constant factor is called the common ratio of the series $a, ar, ar^2, ar^3, \ldots$ is a G.P. with $a$ as the first term and $r$ as common ratio.

$n^{th}$ term of G.P.:

$$T_n = a(r)^{n-1}$$

Sum of the first $n$ terms:

$$S_n = \begin{cases} \frac{a(r^n - 1)}{r-1}, & r \neq 1 \\ \frac{na}{1-r}, & r = 1 \end{cases}$$

Sum of an infinite G.P., when $|r| < 1$.

$$S_\infty = \frac{a}{1-r}$$

$p^{th}$ term from the end of a finite G.P.:

If G.P. consists of 'n' terms, $p^{th}$ term from the end = $(n - p + 1)^{th}$ term from the beginning = $ar^{n-p}$.

Also, the $p^{th}$ term from the end of a G.P. with last term $\ell$ and common ratio $r$ is $\ell \left( \frac{1}{r} \right)^{n-1}$

(5) Properties of G.P.:

(i) If each term of a G.P. be multiplied or divided or raised to power by the same non-zero quantity, the resulting sequence is also a G.P.

(ii) The reciprocal of the terms of a given G.P. form a G.P. with common ratio as reciprocal of the common ratio of the original G.P.

(iii) In a finite G.P., the product of terms equidistant from the beginning and the end is always the same and is equal to the product of the first and last term.

i.e., if $a_1, a_2, a_3, \ldots, a_n$ be in G.P. Then $a_1 a_n = a_2 a_{n-1} = a_3 a_{n-2} = a_4 a_{n-3} = \ldots = a_r . a_{n-r+1}$

(iv) If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.

(v) Three non-zero numbers $a, b, c$ are in G.P. iff $b^2 = ac$.

(vi) Every term (except first term) of a G.P. is the square root of terms equidistant from it.

i.e. $T_r = \sqrt{T_{r-p} \cdot T_{r+p}}$; $[r > p]$.

(vii) If first term of a G.P. of $n$ terms is $a$ and last term is $\ell$, then the product of all terms of the G.P. is $a^n \ell^{n/2}$.

(viii) If there are $n$ quantities in G.P. whose common ratio is $r$ and $S_n$ denotes the sum of the first $m$ terms, then the sum of their product taken two by two is $\frac{r}{r+1} S_n S_{n-1}$. 

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Mathematics Concept Note
IIT-JEE/ISI/CMI
Page 63
(ix) Any three consecutive terms of a G.P. can be taken as \( \frac{a}{r}, a, ar \), in general we take
\[
a, a, ar, \ldots, a, ar^2, \ldots, ar^n
\]
in case we have to take \( 2k + 1 \) terms in a G.P.

(x) Any four consecutive terms of a G.P. can be taken as \( \frac{a}{r^3}, \frac{a}{r^2}, ar, ar^2 \), in general we take
\[
a, a, ar, ar^2, \ldots, ar^{2k-1}
\]
in case we have to take \( 2k \) terms in a G.P.

(xi) If \( a_1, a_2, a_3, \ldots \) and \( b_1, b_2, b_3, \ldots \) are two G.P.'s with common ratio \( r_1 \) and \( r_2 \) respectively then the sequence \( a_1 b_1, a_2 b_2, a_3 b_3, \ldots \) is also a G.P. with common ratio \( r_1 r_2 \).

(xii) If \( a_1, a_2, a_3, \ldots \) are in G.P. where each \( a_i > 0 \), then \( \log a_1, \log a_2, \log a_3, \ldots \) are in A.P. and its converse is also true.

(6) **Geometric Means (G.M.)**:

If \( a, b, c \) are in G.P., \( b \) is the G.M. between \( a \) and \( c \).

\[ b^2 = ac \]

**n-Geometric Means Between \( a, b \):**

If \( a, b \) are two given numbers and \( a, G_1, G_2, \ldots, G_n, b \) are in G.P.. Then \( G_1, G_2, G_3, \ldots, G_n \) are \( n \) G.M.s between \( a \) and \( b \).

\[ G_1 = a\left(\frac{b}{a}\right)^{1/n+1}, \quad G_2 = a\left(\frac{b}{a}\right)^{2/n+1}, \quad \ldots, \quad G_n = a\left(\frac{b}{a}\right)^{n/n+1} \]

**Note:**

The product of \( n \) G.M.s between \( a \) and \( b \) is equal to the \( n \)th power of the single G.M. between \( a \) and \( b \)

\[ \prod_{r=1}^{n} G_r = (G)^n, \text{ where } G \text{ is the single G.M. between } a \text{ and } b. \]

(7) **Harmonic Progression**:

The sequence \( a_1, a_2, \ldots, a_n, \ldots \) where \( a_i \neq 0 \) for each \( i \) is said to be in harmonic progression (H.P.) if the sequence \( \frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}, \ldots \) is in A.P. Note that \( a_n \), the \( n \)th term of the H.P., is given by

\[ a_n = \frac{1}{a + (n - 1)d} \text{ where } a = \frac{1}{a_1} \text{ and } d = \frac{1}{a_2} - \frac{1}{a_1}. \]

**Note:**

(i) If \( a \) and \( b \) are two non-zero numbers, then the harmonic mean of \( a \) and \( b \) is number \( H \) such that the sequence \( a, H, b \) is an H.P. We have

\[ \frac{1}{H} = \frac{1}{n} \left( \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \right) \]

(ii) The \( n \) numbers \( H_1, H_2, \ldots, H_n \) are said to be harmonic means between \( a \) and \( b \) if \( a, H_1, H_2, \ldots, H_n, b \) are in H.P., that is, if \( 1/a, 1/H_1, 1/H_2, \ldots, 1/b \) are in A.P. Let \( d \) be the common difference of this A.P. Then

\[ \frac{1}{b} = \frac{1}{a} + (n + 2 - 1)d \]
\[ d = \frac{1}{n+1} \left( \frac{1}{b} - \frac{1}{a} \right) = \frac{a-b}{(n+1)ab} \]

Thus

\[ \frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{(n+1)ab} \]

\[ \frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{(n+1)ab} , \ldots , \frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab} \]

From here, we can get the values of \( H_1, H_2, \ldots, H_n \).

(8) **Relation between Means:**

If \( A, G, H \) are respectively A.M., G.M., H.M. between \( a \) and \( b \) both being unequal and positive then,

\[ G^2 = AH \Rightarrow A, G, H \text{ are in G.P.} \]

Let \( a_1, a_2, a_3, \ldots, a_n \) be \( n \) positive real numbers, then \( A.M. \geq G.M. \geq H.M. \), where

\[ A.M. = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}, \quad G.M. = (a_1 a_2 a_3 \ldots a_n)^{1/n} \quad \text{and} \]

\[ H.M. = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}. \]

**Note:**

(i) The inequality of A.M., G.M. and H.M. is only applicable for positive real numbers and the sign of equality holds for equal number.

\[ \therefore \quad A.M. = G.M. = H.M. \]

\[ \Rightarrow \quad a_1 = a_2 = a_3 = \ldots = a_n. \]

(ii) If \( a, b, c \in \mathbb{R}^+ \), then \( \frac{a+b+c}{3} \geq (abc)^{1/3} \geq \frac{3abc}{ab+bc+ac} \)

(9) **Arithmetico-Geometric Series:**

A series each term of which is formed by multiplying the corresponding term of an A.P. and G.P. is called the Arithmetico-Geometric Series. e.g. \( 1 + 3x + 5x^2 + 7x^3 + \ldots \)

Here \( 1, 3, 5, \ldots \) are in A.P. and \( 1, x, x^2, \ldots \) are in G.P...

**Sum of n terms of an Arithmetico-Geometric Series:**

Let \( S_n = a + (a+d)r + (a + 2d) r^2 + \ldots + [a + (n-1)d] r^{n-1} \), then

\[ S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d] r^n}{1-r}, \quad r \neq 1. \]

**Sum To Infinity:** If \( |r| < 1 \) and \( n \to \infty \), then \( S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} \).
(10) Important Results:

(i) \[ \sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r \]

(ii) \[ \sum_{r=1}^{n} ka_r = k \sum_{r=1}^{n} a_r \]

(iii) \[ \sum_{r=1}^{n} k = k + k + k \ldots \ldots \ldots n \text{ times} = nk ; \text{ where } k \text{ is a constant} \]

(iv) \[ \sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots \ldots + n = \frac{n(n+1)}{2} \]

(v) \[ \sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \]

(vi) \[ \sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots \ldots + n^3 = \left( \frac{n(n+1)}{2} \right)^2 \]

(vii) \[ 2 \cdot \sum_{1 \leq i < j \leq n} a_i a_j = (a_1 + a_2 + \ldots \ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots \ldots + a_n^2) \]

(viii) \[ \prod_{r=1}^{n} a_r = a_1 a_2 a_3 \ldots \ldots a_n \]

(ix) \[ \prod_{r=1}^{n} ka_r = k^n \cdot \prod_{r=1}^{n} a_r \]

(x) \[ \prod_{r=1}^{n} r = n! \]

(xi) \[ \prod_{r=1}^{n} (a_r b_r) = \left( \prod_{r=1}^{n} a_r \right) \left( \prod_{r=1}^{n} b_r \right) \]

(11) Method of Difference:

If the differences of the successive terms of a series are in A.P. or G.P., we can find the \( n \)th term of the series by the following steps:

Step 1: Denote the \( n \)th term by \( T_n \) and the sum of the series up to \( n \) terms by \( S_n \).

Step 2: Rewrite the given series with each term shifted by one place to the right.

Step 3: By subtracting the later series from the former, find \( T_n \).

Step 4: From \( T_n \) and \( S_n \), \( S_n \) can be found by appropriate summation.
10. FUNCTIONS AND GRAPH

(1) Definition:

Function is a particular case of relation, from a non empty set A to a non empty set B, that associates each and every member of A to a unique member of B. Symbolically, we write f : A → B. We read it as "f is a function from A to B".

Note:
(i) Let f : A → B, then the set A is known as the domain of f & the set B is known as co-
domain of f.
(ii) If a member 'a' of A is associated to the member 'b' of B, then 'b' is called the f-image
of 'a' and we write b = f(a). Further 'a' is called a pre-image of 'b'.
(iii) The set {f(a) : a ∈ A} is called the range of f and is denoted by f(A). Clearly f(A) ⊆ B.
(iv) Sometimes if only definition of f(x) is given (domain and s are not mentioned),
then domain is set of those values of 'x' for which f(x) is defined, while codomain is
considered to be (−∞, ∞).
(v) A function whose domain and range both are sets of real numbers is called a real
function. Conventionally the word "FUNCTION" is used only as the meaning of real
function.

(2) Classification of Functions:

On the basis of mapping functions can be classified as follows:

(i) One-One Function (Injective Mapping)
A function f : A → B is said to be a one-one function or injective mapping if different
elements of A have different f images in B.
Thus for x₁, x₂ ∈ A & f(x₁), f(x₂) ∈ B, f(x₁) = f(x₂) ⇒ x₁ = x₂ or x₁ ≠ x₂ ⇒ f(x₁) ≠ f(x₂).
Diagrammatically an injective mapping can be shown as

(ii) Many - One function :
A function f : A → B is said to be a many one function if two or more elements of A
have the same f image in B.
Thus f : A → B is many one iff there exists at least two elements x₁, x₂ ∈ A , such that
f(x₁) = f(x₂) but x₁ ≠ x₂.
Diagrammatically a many one mapping can be shown as

Note:
(i) If x₁, x₂ ∈ A & f(x₁), f(x₂) ∈ B, f(x₁) = f(x₂) ⇒ x₁ = x₂ or x₁ ≠ x₂ ⇒ f(x₁) ≠ f(x₂), then
function is ONE-ONE otherwise MANY-ONE.
(ii) If there exists a straight line parallel to x-axis, which cuts the graph of the function
atleast at two points, then the function is MANY-ONE, otherwise ONE-ONE.
(iii) If either f'(x) ≥ 0, ∀ x ∈ complete domain or f'(x) ≤ 0 ∀ x ∈ complete domain, where
equality can hold at discrete point(s) only, then function is ONE-ONE, otherwise
MANY-ONE.
(iii) **Onto function (Surjective mapping)**

If the function \( f : A \rightarrow B \) is such that each element in \( B \) (co-domain) must have at least one pre-image in \( A \), then we say that \( f \) is a function of \( A \) 'onto' \( B \). Thus \( f : A \rightarrow B \) is surjective iff \( \forall b \in B \), there exists some \( a \in A \) such that \( f(a) = b \).

Diagrammatically surjective mapping can be shown as

\[
\begin{array}{c}
A \quad \text{OR} \quad B
\end{array}
\]

**Note:** If range \( \equiv \) co-domain, then \( f(x) \) is onto, otherwise into

(iv) **Into function :**

If \( f : A \rightarrow B \) is such that there exists at least one element in co-domain which is not the image of any element in domain, then \( f(x) \) is into.

Diagrammatically into function can be shown as

\[
\begin{array}{c}
A \quad \text{OR} \quad B
\end{array}
\]

**Note:**

(i) function can be one of following four types :

(a) one-one onto (injective & surjective)

(b) one-one into (injective but not surjective)

(c) many-one onto (surjective but not injective)

(d) many-one into (neither surjective not injective)

(ii) If \( f \) is both injective & surjective, then it is called a bijective mapping. The bijective functions are also named as invertible, non-singular or biuniform functions.

(iii) If a set \( A \) contains 'n' distinct elements then the number of different functions defined from \( A \rightarrow A \) is \( n^n \) and out of which \( n! \) are one one.

(3) **Various Types of Functions :**

(i) **Constant function :**

A function \( f : A \rightarrow B \) is said to be a constant function if every element of \( A \) has the same image in \( B \). Thus \( f : A \rightarrow B \); \( f(x) = c, \forall x \in A, c \in B \) is a constant function.

**Note:**

(i) Constant function is even and \( f(x) = 0 \) is the only function which is both even and odd

(ii) Constant function is periodic with indeterminate periodicity.
(ii) **Identity function** :
The function \( f : A \to A \) defined by \( f(x) = x \ \forall \ x \in A \) is called the identity function on \( A \) and is denoted by \( I_A \). It is easy to observe that identity function is a bijection.

![Identity function graph](image)

(iii) **Polynomial function** :
If a function \( f \) is defined by \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) where \( n \) is a non-negative integer and \( a_n, a_{n-1}, \ldots, a_0 \) are real numbers and \( a_n \neq 0 \), then \( f \) is called a polynomial function of degree \( n \).

For example: linear functions, quadratic functions, cubic functions etc.

(iv) **Algebraic Function** :
\( y \) is an algebraic function of \( x \), if it is a function that satisfies an algebraic equation of the form, \( P_0(x)y^n + P_1(x)y^{n-1} + \ldots + P_{n-1}(x)y + P_n(x) = 0 \) where \( n \) is a positive integer and \( P_0(x), P_1(x), \ldots \) are polynomials in \( x \). e.g. \( y = |x| \) is an algebraic function, since it satisfies the equation \( y^2 - x^2 = 0 \).

**Note:**
(i) All polynomial functions are algebraic but not the converse.
(ii) A function that is not algebraic is called **Transcendental Function**.

(v) **Fractional/Rational Function** :
A rational function is a function of the form \( y = f(x) = \frac{g(x)}{h(x)} \), where \( g(x) \) & \( h(x) \) are polynomials and \( h(x) \neq 0 \).

(vi) **Exponential Function** :
A function \( f(x) = a^x = e^{\ln a}x \) \( (a > 0, \ a \neq 1, \ x \in \mathbb{R}) \) is called an exponential function.

Graph of exponential function can be as follows:

For \( a > 1 \)

![Exponential function graph for a > 1](image)

For \( 0 < a < 1 \)

![Exponential function graph for 0 < a < 1](image)

(vii) **Logarithmic Function** :
\( f(x) = \log_a x \) is called logarithmic function where \( a > 0 \) and \( a \neq 1 \) and \( x > 0 \). Its graph can be as follows:

For \( a > 1 \)

![Logarithmic function graph for a > 1](image)

For \( 0 < a < 1 \)

![Logarithmic function graph for 0 < a < 1](image)
(viii) Absolute Value Function / Modulus Function:

The symbol of modulus function is \( f(x) = |x| \) and is defined as:
\[
y = |x| = \begin{cases} 
x & \text{if } x \geq 0 \\
-x & \text{if } x < 0
\end{cases}
\]

Note:

(i) \( \sqrt{x^2} = |x| \) and \( |x| = \max\{x, -x\} \)

(ii) Absolute value function is piece-wise defined function and its domain and range are \( \mathbb{R} \) and \([0, \infty)\) respectively

(iii) If 'a' and 'b' are non-negative real numbers, then

\[
\begin{align*}
|ax| &\leq |a||x| \quad \text{and} \quad |x| \max\{a, -a, x\} \\
|ax| &\geq |a||x| \quad \text{and} \quad |x| \max\{-a, -x\} \\
\end{align*}
\]

(iv) \( \max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} \) if \( f(x) \geq g(x) \)

(v) \( \min\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2} \)

(ix) Signum Function:

A function \( f(x) = \text{sgn} \ (x) \) is defined as follows:
\[
f(x) = \text{sgn} \ (x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0
\end{cases}
\]

Note:

(i) \( \text{sgn} \ x = \begin{cases} 
|x| & x \neq 0 \\
x & x = 0
\end{cases} \)

(ii) \( \text{sgn} \ f(x) = \begin{cases} 
\frac{|f(x)|}{f(x)} & f(x) \neq 0 \\
0 & f(x) = 0
\end{cases} \)

(x) Greatest Integer Function or Step Up Function/Box function:

The function \( y = f(x) = \lfloor x \rfloor \) is called the greatest integer function where \( \lfloor x \rfloor \) equals to the greatest integer less than or equal to \( x \).

Note:

(i) \( \lfloor x \rfloor = x \forall x \in \mathbb{I} \)

(ii) \( \lfloor x + n \rfloor = \lfloor x \rfloor + n \forall n \in \mathbb{I} \)

(iii) \( x - 1 < \lfloor x \rfloor \leq x \)

(iv) \( \lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 
0 & x \in \mathbb{I} \\
-1 & x \notin \mathbb{I}
\end{cases} \)

(v) \( \lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \)
(xi) **Fractional Part Function:**

It is defined as, \( y = \{x\} = x - [x] \) and \( \{x\} \in [0,1) \)

**Note:**

\( f(x) = x - [x], \sqrt{x - [x]} \) or \( (x - [x])^2 \) are periodic with period 1.

(4) **Trigonometric Function:**

(i) \( y = \sin x \); \( x \in \mathbb{R}; y \in [-1, 1] \)

(ii) \( y = \cos x \); \( x \in \mathbb{R}; y \in [-1, 1] \)

(iii) \( y = \tan x \); \( x \in \mathbb{R} - (2n + 1)\pi/2, n \in \mathbb{I}; y \in \mathbb{R} \)

(iv) \( y = \cot x \); \( x \in \mathbb{R} - n\pi, n \in \mathbb{I}; y \in \mathbb{R} \)

(v) \( y = \csc x \);

\( x \in \mathbb{R} - n\pi, n \in \mathbb{I}; y \in \mathbb{R}/(-1, 1) \)

(vi) \( y = \sec x \);

\( x \in \mathbb{R} - (2n + 1)\pi/2, n \in \mathbb{I}; y \in \mathbb{R}/(-1, 1) \)

(5) **Inverse Trigonometric Function:**

(i) \( y = \sin^{-1} x, \ |x| \leq 1, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \)

(ii) \( y = \cos^{-1} x, \ |x| \leq 1, y \in [0, \pi] \)
(iii) \( y = \tan^{-1} x, \ x \in \mathbb{R}, \ y \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) 

(iv) \( y = \cot^{-1} x, \ x \in \mathbb{R}, \ y \in (0, \pi) \)

(v) \( y = \sec^{-1} x, \ |x| \geq 1, \ y \in \left[ 0, \frac{\pi}{2} \right] \cup \left( \frac{\pi}{2}, \pi \right) \) 

(vi) \( y = \csc^{-1} x, \ |x| \geq 1, y \in \left[ -\frac{\pi}{2}, 0 \right] \cup \left( 0, \frac{\pi}{2} \right) \)

(6) **Equal or Identical Function**

Two functions \( f(x) \) and \( g(x) \) are said to be identical (or equal) iff:

(i) The domain of \( f \equiv \) the domain of \( g \).
(ii) The range of \( f \equiv \) the range of \( g \) and

(iii) \( f(x) = g(x) \), for every \( x \) belonging to their common domain, e.g. \( f(x) = \frac{1}{x} \) and 

\[
g(x) = \frac{x}{x^2}
\]

are identical functions. But \( f(x) = x \) and \( g(x) = \frac{x^2}{x} \) are not identical functions.

(7) **Bounded Function**

The function \( f(x) \) is said to be bonded above if there exists \( M \) such that \( y = f(x) \leq M \) (i.e. not greater than \( M \)) for all \( x \) of the domain and \( M \) is called upper bound. Similarly \( f(x) \) is said to be bounded below if there exists in such that \( y = f(x) \geq m \) (i.e. never less than \( m \)) for all \( x \) of the domain and \( m \) is called the lower bound.

If however, there does not exist \( M \) and \( m \) as stated above, the function is said to be unbounded.

(8) **Even and odd Functions**

- If \( f(-x) = f(x) \) for all \( x \) in the domain of \( f \) then \( f \) is said to be an even function.
- If \( f(x) - f(-x) = 0 \Rightarrow f(x) \) is even.
If \( f(-x) = -f(x) \) for all \( x \) in the domain of \( f \) then \( f \) is said to be an odd function.

If \( f(x) + f(-x) = 0 \) \( \Rightarrow \) \( f(x) \) is odd.

**Note:**

(i) Even functions are symmetrical about the \( y \)-axis and odd functions are symmetrical about origin.

(ii) Any given function can be uniquely expressed as the sum of even function and odd function for example:

\[
 f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]
\]

(iii) For any given function \( f(x) \), \( g(x)=f(x)+f(-x) \) is even function and \( h(x)=f(x)-f(-x) \) is odd function.

(iv) If an odd function is defined at \( x = 0 \), then \( f(0) = 0 \).

(v) Let the definition of the function \( f(x) \) is given only for \( x \geq 0 \). Even extension of this function implies to define the function for \( x < 0 \) assuming it to be even. In order to get even extension replace \( x \) by \( -x \) in the given definition. Similarly, odd extension implies to define the function for \( x < 0 \) assuming it to be odd. In order to get odd extension, multiply the definition of even extension by \(-1\).

(9) **Periodic Function:**

A function \( f(x) \) is called periodic with a period \( T \) if there exists a positive real number \( T \) such that for each \( x \) in the domain of \( f \) the numbers \( x - T \) and \( x + T \) are also in the domain of \( f \) and \( f(x) = f(x + T) \) for all \( x \) in the domain of \( f \). Domain of a periodic function is always unbounded. Graph of a periodic function with period \( T \) is repeated after every interval of \( T \).

The least positive period is called the **principal or fundamental period** of \( f \) or simply the period of \( f \).

**Note:**

(i) \( \sin^n, \cos^n, \sec^n, \) and \( \cosec^n x \) are periodic function with \( \pi \) period when \( n \) is even and \( 2\pi \) when \( n \) is odd or fraction. e.g. period of \( \sin^2 x \) is \( \pi \) but period of \( \sin^3 x \), \( \sqrt{\sin x} \) is \( 2\pi \).

(ii) \( \tan^n x \) and \( \cot^n x \) are periodic functions with period \( \pi \) irrespective of \( 'n' \).

(iii) \( |\sin x|, |\cos x|, |\tan x|, |\cot x|, |\sec x|, \) and \( |\cosec x| \) are periodic functions with period \( \pi \).

(iv) If \( f(x) \) is periodic with period \( T \), then:

a) \( k \cdot f(x) \) is periodic with period \( T \).

b) \( f(x+b) \) is periodic with period \( T \).

c) \( f(x)+c \) is periodic with period \( T \).

(v) If \( f(x) \) has a period \( T \), then \( \frac{1}{f(x)} \) and \( \sqrt{f(x)} \) also have a period \( T \).

(vi) If \( f(x) \) has a period \( T \) then \( f(ax + b) \) has a period \( \frac{T}{|a|} \).

(vii) If \( f(x) \) has a period \( T_1 \) and \( g(x) \) has a period \( T_2 \) then period of \( f(x) \pm g(x) \) or \( f(x)g(x) \)

\[
g(x) \text{ or } \frac{f(x)}{g(x)} \text{ is L.C.M of } T_1 \& T_2 \text{ provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists } f(x) \pm g(x) \text{ or } f(x).g(x) \text{ is not periodic.}
\]

e.g. \( |\sin x| \) has the period \( \pi \), \( |\cos x| \) also has the period \( \pi \).

\[
|\sin x| + |\cos x| \text{ also has a period } \pi . \text{ But the fundamental period of } |\sin x| + |\cos x| \text{ is } \frac{\pi}{2}.
\]
(10) Composite Function:

Let \( f : X \rightarrow Y_1 \) and \( g : Y_2 \rightarrow Z \) be two functions and the set \( D = \{ x \in X : f(x) \in Y_2 \} \). If \( D \neq \emptyset \) then the function \( h \) defined on \( D \) by \( h(x) = g(f(x)) \) is called composite function of \( g \) and \( f \) and is denoted by \( gof \). It is also called function of a function.

**Note:**
(i) Domain of \( gof \) is \( D \) which is a subset of \( X \) (the domain of \( f \)). Range of \( gof \) is a subset of the range of \( g \). If \( D = X \), then \( f(x) \subseteq Y_2 \).
(ii) In general \( gof \neq fog \) (i.e. not commutative)
(iii) The composite of functions are associative i.e. if three functions \( f, g, h \) are such that \( fo(goh) \) and \( (fog)oh \) are defined, then \( fo(goh) = (fog)oh \).
(iv) If \( f \) and \( g \) both are one-one, then \( gof \) and \( fog \) would also be one-one.
(v) If \( f \) and \( g \) both are onto, then \( gof \) or \( fog \) may or may not be onto.
(vi) The composite of two bijections is a bijection iff \( f \) & \( g \) are two bijections such that \( gof \) is defined, then \( gof \) is also a bijection only when co-domain of \( f \) is equal to the domain of \( g \).
(vii) If \( g \) is a function such that \( gof \) is defined on the domain of \( f \) and \( f \) is periodic with \( T \), then \( gof \) is also periodic with \( T \) as one of its periods.

(11) Inverse of a Function:

Let \( f : A \rightarrow B \) be a function, then \( f \) is invertible iff there is a function \( g : B \rightarrow A \) such that \( gof(x) \) is an identity function on \( A \) and \( fog(x) \) is an identity function on \( B \), \( g(x) \) is called inverse of \( f \) and is denoted by \( f^{-1} \). For a function to be invertible it must be bijective.

**Note:**
(i) The graphs of \( f \) & \( g \) are the mirror images of each other in the line \( y = x \). For example \( f(x) = ax \) and \( g(x) = \log_a x \) are inverse of each other, and their graphs are mirror images of each other on the line \( y = x \) as shown below.

![Graph of \( f(x) = ax, a > 1 \) and \( g(x) = \log_a x \)](image)

(ii) Normally points of intersection of \( f \) and \( f^{-1} \) lie on the straight line \( y = x \). However it must be noted that \( f(x) \) and \( f^{-1}(x) \) may intersect otherwise also.
(iii) In general \( fog(x) \) and \( gof(x) \) are not equal but if they are equal then in majority of cases \( f \) and \( g \) are inverse of each other or atleast one of \( f \) and \( g \) is an identity function.
(iv) If \( f \) & \( g \) are two bijections \( f : A \rightarrow B \), \( g : B \rightarrow C \) then the inverse of \( gof \) exists and \( (gof)^{-1} = f^{-1} \circ g^{-1} \).
(v) If \( f(x) \) and \( g(x) \) are inverse function of each other then \( f(g(x)) = \frac{1}{g'(x)} \).
(12) Functional relationships:

If $x$, $y$ are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, n \in \mathbb{R}$

(iii) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^x$.

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where $k$ is a constant.

(v) $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \Rightarrow f(x) = 1 \pm x^n$ where $n \in \mathbb{N}$.

(vi) $f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2} \Rightarrow f(x) = ax + b$, $a$ and $b$ are constants.
11. LIMITS

(1) Definition:

Let \( f(x) \) be a function, if for every positive number \( \varepsilon \) there exists a positive number \( \delta \), such that \( 0 < |x - a| < \delta \) and \( |f(x) - L| < \varepsilon \), then \( f(x) \) tends to limit \( L \) as \( x \) tends to \( a \) and the limiting value of \( f(x) \) at location \( x = a \) is represented by \( \lim_{x \to a} f(x) \).

- In above definition of limit if \( x \) tends to \( a \) from the values of \( x \) greater than \( a \), then limiting value is termed as right hand limit (RHL.) and represented by \( \lim_{x \to a^+} f(x) \) or \( f(a^+) \).

\[ f(a^+) = \lim_{h \to 0^+} f(a + h) \] where \( h \) is infinitely small positive number.

- In above definition of limit if \( x \) tends to \( a \) from the values of \( x \) less than \( a \), then limiting value is termed as left hand limit (LHL.) and represented by \( \lim_{x \to a^-} f(x) \) or \( f(a^-) \).

\[ f(a^-) = \lim_{h \to 0^-} f(a - h) \] where \( h \) is infinitely small positive number.

Note:

(i) Finite limit of a function \( f(x) \) is said to exist as \( x \) approaches \( a \), if:

\[ \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \text{some finite value}. \]

(ii) If LHL and RHL both approaches to \( \infty \) or \( -\infty \), then limit of function is said to be infinite limit.

(2) Indeterminate Forms:

Let \( f(x) \) be a function which is defined is same neighbourhood of location \( x = a \), but functioning value \( f(a) \) is indeterminate because of the form of indeterminacy

\[ \left( \text{i.e. } \frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty^0, 0^0 \text{ and } 1^\infty \right). \]

For example:

\[ f(x) = \frac{x^2 - 4}{x - 2} \text{ is at } x = 2, \quad f(x) = \frac{1}{x^2} - \frac{1}{\sin^2 x} \text{ at } x = 0, \quad f(x) = (1 - \cos x)^x \text{ at } x = 0, \text{ etc.} \]

To know the behavior of function at indeterminate locations, limiting values are calculated, which is represented by \( \lim_{x \to a} f(x) \), where \( x = a \) is the location of indeterminacy for function \( f(x) \).

Note:

If function \( f(x) \) is determinate at locations \( x = a \), then \( \lim_{x \to a} f(x) = f(a) \), provided the limit exists (i.e. RHL = LHL).

For example: If \( f(x) = \cos x \), then \( \lim_{x \to 0} f(x) = f(0) \)

- If \( f(x) = \frac{x^2 - 4}{x - 2} \), then \( \lim_{x \to 2} f(x) \neq f(2) \).

(3) Fundamental Theorems on Limits:

Let \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \). If \( L \) and \( M \) exists, then:
(i) \[ \lim_{x \to a} \{f(x) \pm g(x)\} = L + M \quad \{L \pm M \neq \infty - \infty\} \]

(ii) \[ \lim_{x \to a} \{f(x) \cdot g(x)\} = LM \quad \{LM \neq 0 \times \infty\} \]

(iii) \[ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{provided } m \neq 0 \quad \left\{ \begin{array}{l} L < 0 \quad \text{or} \quad L \to \infty \quad \text{or} \quad L \to -\infty \end{array} \right. \]

(iv) \[ \lim_{x \to a} (f(x))^g(x) = (L)^M \quad \left\{ \begin{array}{l} (L)^M \neq 0^0 \quad \text{or} \quad (L)^M \to \infty \quad \text{or} \quad (L)^M \to 1^1 \end{array} \right. \]

(v) \[ \lim_{x \to a} k f(x) = k \lim_{x \to a} f(x) \quad \text{where } k \text{ is a constant.} \]

(vi) \[ \lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) = f(M) \quad \text{provided } f \text{ is continuous at } \lim_{x \to a} g(x) = M. \]

(vii) \[ \lim_{x \to a} \log f(x) = \{\lim_{x \to a} f(x)\} \]

(viii) \[ \lim_{x \to a} \frac{d}{dx} (f(x)) = \frac{d}{dx} \left( \lim_{x \to a} f(x) \right) \]

(ix) If \[ \lim_{x \to a} f(x) = l \quad \text{or} \quad -\infty \quad \text{or} \quad \infty, \quad \text{then} \quad \lim_{x \to a} \frac{1}{f(x)} = 0 \]

(4) **Standard Limits:**

(a) If 'x' is measured in radians, then \[ \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{\tan x}{x} = 1 \]

(b) \[ \lim_{x \to 0} \frac{\cos x}{x} = 0 \]

(c) \[ \lim_{x \to 0} \frac{x^n - a^n}{x - a} = n(a)^{n-1} \]

(d) \[ \lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1 \]

(e) \[ \lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1 \]

(f) \[ \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \]

(g) \[ \lim_{x \to 0} \frac{a^x - 1}{x} = \log_a a \]

(h) \[ \lim_{x \to 0} \frac{\log_a (1 + x)}{x} = 1 \]

(i) \[ \lim_{x \to 0} \frac{\log_a (1 + x)}{x} = \frac{\log_a e}{x} \]

(j) If \( m, n \in \mathbb{N} \) and \( a_0, b_0 \neq 0 \), then

\[ \lim_{x \to a} \frac{a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n}{b_0 x^m + b_1 x^{m-1} + b_2 x^{m-2} + \ldots + b_m} = \begin{cases} \frac{a_0}{b_0} & ; m = n \\ 0 & ; n < m \\ \infty & ; n > m \end{cases} \]

(k) \[ \lim_{x \to \infty} a^x = \begin{cases} 1 & ; a = 1 \\ 0 & ; |a| < 1 \\ \infty & ; a > 1 \end{cases} \quad \text{oscillates finitely} \quad \text{oscillates infinitely} \quad \text{oscillates infinitely} \quad \text{oscillates infinitely} \]

(l) \[ \lim_{x \to 0} (1 + x)^{1/x} = e \]
\[
\lim_{x \to a} \left( \frac{1 + \frac{1}{x}}{x} \right)^x = e
\]

\[
\lim_{x \to a} (f(x))^{g(x)} = e^{\lim_{x \to a} g(x) \log_e(f(x))}
\]

(o) If \( (f(x))^{g(x)} \to 1^\infty \) for \( x \to a \), then \( \lim_{x \to a} (f(x))^{g(x)} = e^{\lim_{x \to a} (f(x)-1)g(x)} \).

(5) Limits Using Expansion

Let \( f(n) \) be a function which is differentiable for all orders throughout some interval containing location \( x = 0 \) as an interior point, then Taylor's series generated by \( f(x) \) at \( x = 0 \) (i.e. Maclaurin's series) is given by:

\[
f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots
\]

For example: If \( f(x) = e^x \), then \( e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \)

In solving limits sometimes standard series are more convenient than standard methods and hence following series should be remembered.

(i) \( e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \)

(ii) \( a^x = 1 + \frac{\ln a}{1!}x + \frac{\ln^2 a}{2!}x^2 + \frac{\ln^3 a}{3!}x^3 + \ldots \) \( (a > 0, a \neq 1) \)

(iii) \( \ln (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \) \( (-1 < x < 1) \)

(iv) \( \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \)

(v) \( \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \)

(vi) \( \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \)

(vii) \( \tan^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \)

(viii) \( \sin^{-1} x = x + \frac{x^3}{3!} + \frac{x^5}{5!} x^2 + \frac{1 \cdot 3 \cdot 5^2}{7!} x^3 + \ldots \)

(ix) \( \sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots \)

(x) \( (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \) \( (|x| < 1, n \in \mathbb{R}) \)

(6) L' Hospital's Rule:

Let \( f(x) \) and \( g(x) \) are differentiable function of \( x \) at location \( x = a \) such that:

\[
\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \quad \text{or} \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty
\]

Now, according to L' Hospital Rule

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]
Note:

If \( \lim_{x \to a} \frac{f'(x)}{g'(x)} \) assumes the indeterminate form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \) and \( f'(x) \), \( g'(x) \) satisfy the conditions for L'Hôpital rule then application of the rule can be repeated.

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}.
\]

(7) Sandwich Theorem (Squeeze Play Theorem):

Let \( f(x) \), \( g(x) \) and \( h(x) \) be functions of \( x \) such that \( g(x) \leq f(x) \leq h(x) \) \( \forall x \in (a-h, a+h) \), where 'h' is infinitely small positive number, then \( \lim_{x \to a} g(x) \leq \lim_{x \to a} f(x) \leq \lim_{x \to a} h(x) \).

Now, if \( \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L \Rightarrow \lim_{x \to a} f(x) = L \).
12. CONTINUITY AND DIFFERENTIABILITY

(1) Continuity of a function:
A function \( f(x) \) is said to be continuous at \( x = c \), if:
\[
\lim_{{x \to c}} f(x) = f(c)
\]
Geometrically, if a function \( f(x) \) is continuous at \( x = c \), the graph of \( f(x) \) at the corresponding point \((c, f(c))\) will not be broken, but if \( f(x) \) is discontinuous at \( x = c \), the graph will have a break at the corresponding point.
A function \( f(x) \) can be discontinuous at \( x = c \) because of any of the following three reasons:

(i) \( \lim_{{x \to c}} f(x) \) does not exist (i.e., \( \lim_{{x \to c^-}} f(x) \neq \lim_{{x \to c^+}} f(x) \))
(ii) \( f(x) \) is not defined at \( x = c \)
(iii) \( \lim_{{x \to c}} f(x) \neq f(c) \)

Following graph illustrates the different cases of discontinuity of \( f(x) \) at \( x = c \).

(2) Types of Discontinuity:

(a) Discontinuity of First Kind:
Let \( f(x) \) be a function for which left hand limit and right hand limit at location \( x = a \) are finitely existing. Now if \( f(x) \) is discontinuous at \( x = a \) is termed as discontinuity of first kind which may be removable or non-removable.

Removable Discontinuity of 1st kind:
If \( \lim_{{x \to a}} f(x) \) exists but is not equal to \( f(a) \), then the function is said to have a removable discontinuity at \( x = a \). In this case function can be redefined such that \( \lim_{{x \to a}} f(x) = f(a) \) and hence function can become continuous at \( x = a \).

Removable type of discontinuity of 1st kind can be further classified as:
**Missing Point Discontinuity:**

Where \( \lim_{x \to a} f(x) \) exists finitely but \( f(a) \) is not defined.

For example: \( f(x) = \frac{x^2 - 4}{x - 2} \) has a missing point discontinuity at \( x = 2 \).

\( f(x) = \frac{\sin x}{x} \) has a missing point discontinuity at \( x = 0 \).

**Isolated Point Discontinuity:**

At locations \( x = a \) where \( \lim_{x \to a} f(x) \) exists and \( f(a) \) also exists but \( \lim_{x \to a} f(x) \neq f(a) \).

For example: 

\[
\begin{align*}
  f(x) &= \begin{cases} 
    x^2/3 & ; x \neq 3 \\
    10 & ; x = 3 
  \end{cases} \\
  f(x) &= [x] + [-x] = \begin{cases} 
    0 & ; x \in \mathbb{I} \\
    -1 & ; x \notin \mathbb{I} 
  \end{cases}
\end{align*}
\]

is discontinuous at integral \( x \).

**Non-Removable discontinuity of I\(^{st}\) Kind:**

If LHL and RHL exists for function \( f(x) \) at \( x = a \) but LHL \( \neq \) RHL, then it is not possible to make the function continuous by redefining it and this discontinuous nature is termed as non-removable discontinuity of I\(^{st}\) kind.

For example: 

(i) \( f(x) = \text{sgn}(x) \) at \( x = 0 \)

(ii) \( f(x) = x - [x] \) at all integral \( x \).

**Note:**

- In case of non-removable discontinuity of the first kind the non-negative difference between the value of the RHL at \( x = a \) and LHL at \( x = a \) is called the jump of discontinuity. Jump of discontinuity = \( |\text{RHL} - \text{LHL}| \)
- A function having a finite number of jumps in a given interval is called a Piece Wise Continuous or Sectionally Continuous function in this interval.

For example: \( f(x) = \{x\} \), \( f(x) = [x] \).

(b) **Discontinuity of Second Kind:**

Let \( f(x) \) be a function for which atleast one of the LHL or RHL is non-existent or infinite at location \( x = a \), then nature of discontinuity at \( x = a \) is termed as discontinuity of second kind and this is non-removable discontinuity.

**Infinite discontinuity:**

At location \( x = a \), either LHL or RHL or both approaches to \( \infty \) or \( -\infty \).

For example: 

\( f(x) = \frac{1}{x-3} \) or \( f(x) = \frac{1}{(x-3)^2} \) at \( x = 3 \).

**Oscillatory discontinuity:**

At location \( x = a \), function oscillates rigorously and can not have a finite limiting.

For example: 

\( f(x) = \sin \left( \frac{1}{x} \right) \) at \( x = 0 \)

**Note:**

Point functions which are defined at single point only are discontinuous of second kind.

For example: 

\( f(x) = \sqrt{x-4} + \sqrt{x-4} \) at \( x = 4 \).
(3) **Continuity in an Interval:**

(a) A function \( f(x) \) is said to be continuous in open interval \((a,b)\) if \( f(x) \) is continuous at each and every interior point of \((a,b)\).

For example: \( f(x) = \tan x \) is continous in \( \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \).

(b) A function \( f(x) \) is said to be continuous in a closed interval \([a,b]\) if :

- \( f(x) \) is continuous in the open interval \((a,b)\) and
- \( f(x) \) is left continuous at \( x = a \)  \( \lim_{x \to a^-} f(x) = f(a) = a \) finite quantity.
- \( f(x) \) is right continuous at \( x = b \)  \( \lim_{x \to b^+} f(x) = f(b) = a \) finite quantity.

(4) **Continuity of Composition of Functions:**

(a) If \( f \) and \( g \) are two functions which are continuous at \( x = c \) then the functions defined by \( G(x) = f(x) \pm g(x) \); \( H(x) = Kf(x) \); \( F(x) = f(x).g(x) \) are also continuous at \( x = c \), where \( K \) is any real number. If \( g(c) \) is not zero , then \( Q(x) = \frac{f(x)}{g(x)} \) is also continuous at \( x = c \).

(b) If \( f(x) \) is continuous and \( g(x) \) is discontinuous at \( x = a \) then the product function \( \phi(x) = f(x).g(x) \) may be continuous but sum or difference function \( \psi(x) = f(x) \pm g(x) \) will necessarily be discontinuous at \( x = a \).

For example: \( f(x) = x \) and \( g(x) = \begin{cases} \sin \frac{x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \).

(c) If \( f(x) \) and \( g(x) \) both are discontinuous at \( x = a \) then the product function \( \phi(x) = f(x).g(x) \) is not necessarily be discontinuous at \( x = a \).

For example: \( f(x) = g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \).

(d) If \( f \) is continuous at \( x = c \) and \( g \) is continuous at \( x = f(c) \) then the composite \( g(f(x)) \) is continuous at \( x = c \).

For example: \( f(x) = \frac{x \sin x}{x^2 + 2} \) and \( g(x) = |x| \) are continuous at \( x = 0 \), hence the composite \( (gof)(x) = \frac{x \sin x}{x^2 + 2} \) will also be continuous at \( x = 0 \).

(5) **Differentiability of a function at a point:**

Let \( y = f(x) \) be a function and points \( P(a, f(a)) \) and \( Q(a + h, f(a + h)) \) lies on the curve of \( f(x) \).

Now , slope of secant passing through \( P \) and \( Q = \frac{f(a + h) - f(a)}{h} \).
If positive scalar h tends to zero, then point Q approaches to point P from right hand side and the secant becomes tangent to curve f(x).

\[ \text{Slope of tangent at } P = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a^+) \]

\( f'(a^+) \) represents the right hand differentiation of function f(x) at location x = a.

Similar to the above discussion, if P(a, f(a)) and R(a – h, f(a–h)) are two points on the curve y = f(x), then slope of secant through P and R = \( \frac{f(a - h) - f(a)}{-h} \).

Now, if positive scales h tends to zero, then point R approaches to point P from left hand side and the secant becomes tangent to curve f(x).

\[ \text{Slope of tangent at } P = \lim_{h \to 0} \frac{f(a - h) - f(a)}{-h} = f'(a^-) \]

\( f'(a^-) \) represents the left hand differentiation of function f(x) at location x = a.

Conditions for differentiability of function at location x = a:

(i) **Necessary condition:** function must be continuous at x = a.

\[ \lim_{x \to a} f(x) = f(a) \] \hspace{1cm} \text{...(1)}

(ii) **Sufficient condition:** function must have finitely equal left hand and right hand derivative. (i.e. \( f'(a^-) = f'(a^+) \))

\[ \lim_{h \to 0} \frac{f(a - h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \text{some finite quantity.} \]

Differentiability of a function at x = a be confirmed if it satisfy both the conditions (i.e. necessary condition and sufficient condition).

**Note:**
- If function is differentiable at a location than it is always continuous at that location but the converse is not always true. Hence all differentiable function are continuous but all continuous functions are not differentiable.
- For example: \( f(x) = e^{-|x|} \) is continous at \( x = 0 \) but not differentiable at \( x = 0 \).
- If \( f(x) \) is differentiable at location \( x = c \), then alternative formula for calculating differentiation is given by:

\[ f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \]
(6) Geometrical Interpretation of Differentiation:

Geometrically a function is differentiable at $x = a$ if there exists a unique tangent of finite slope at location $x = a$.

Non-differentiability of a function $y = f(x)$ at location $x = a$ can be visualized geometrically in the following cases:

(i) Discontinuity in the graph of $f(x)$ at location $x = a$.

(ii) A corner or sharp change in the curvature of graph at $x = a$, where the left hand and right derivatives differ.

(iii) A cusp, where the slope of tangent at $x = a$ approaches $\infty$ from one side and $-\infty$ from the other side.

(iv) Location of vertical tangent at $x = a$, where the slope of tangent approaches $\infty$ from both side or $-\infty$ from both sides.

Note:

- If the graph of a function is smooth curve or gradual curve without any sharp corner (or kink) then also it may be non-differentiable at some location.
  For example: $f(x) = x^{1/3}$ represents a gradual curve without any sharp corner but it is non-differentiable at $x = 0$.
- If a function is non-differentiable at some location then also it may have a unique tangent.
For example: \( y = \text{sgn}(x) \) is discontinuous at \( x = 0 \) and hence non-differentiable but have a unique vertical tangent at \( x = 0 \).

- Geometrically tangent is defined as a line which joins two closed points on the curve.

(7) Differentiable Over an Interval:

\( f(x) \) is said to be differentiable over an open interval if it is differentiable at each and every point of the interval and \( f(x) \) is said to be differentiable over a closed interval \([a,b]\) if \( f(x) \) is continuous in \([a,b]\) and:

(i) for the boundary points \( f'(a^+) \) and \( f'(b^-) \) exist finitely , and

(ii) for any point \( c \) such that \( a < c < b \), \( f'(c^+) \) and \( f'(c^-) \) exist finitely and are equal.

(8) Differentiability of Composition of Functions

(i) If \( f(x) \) and \( g(x) \) are differentiable at \( x = a \) then the functions \( \pm f(x)g(x) \), \( f(x)g(x) \) will also be differentiable at \( x = a \) and if \( g(a) \neq 0 \) then the function \( f(x)/g(x) \) will also be differentiable at \( x = a \).

(ii) If \( f(x) \) is not differentiable at \( x = a \) and \( g(x) \) is differentiable at \( x = a \), then the product function \( f(x)g(x) \) may be differentiable at \( x = a \)

For example: \( f(x) = |x| \) and \( g(x) = x^2 \).

(iii) If \( f(x) \) and \( g(x) \) both are not differentiable at \( x = a \) then the product function \( f(x)g(x) \) may be differentiable at \( x = a \)

For example: \( f(x) = |x| \) and \( g(x) = |x| \).

(iv) If \( f(x) \) and \( g(x) \) both are non-differentiable at \( x = a \) then the sum function \( f(x) \pm g(x) \) may be a differentiable function.

For example: \( f(x) = |x| \) & \( g(x) = -|x| \).
13. DIFFERENTIAL COEFFICIENTS

(1) **First Principle of Differentiation (ab-initio Method):**

- The derivative of a given function \( y = f(x) \) at a point \( x = a \) on its domain is represented by \( f'(a) \) or \( \frac{dy}{dx} \), provided the limit exists, alternatively the derivative of \( y = f(x) \) at \( x = a \) is given by \( f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \), provided the limit exists.

- If \( x \) and \( x + h \) belong to the domain of function \( y = f(x) \), then derivative of function is represented by \( f'(x) \) or \( \frac{dy}{dx} \) and \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \), provided the limit exists.

(2) **Differentiation of Some elementary functions**

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^n )</td>
<td>( nx^{n-1} )</td>
</tr>
<tr>
<td>( a^x )</td>
<td>( a^x \ln a )</td>
</tr>
<tr>
<td>( \ln</td>
<td>x</td>
</tr>
</tbody>
</table>
| \( \log_ax \)| \( \frac{1}{x} \log_a e \)
| \( \sin x \)  | \( \cos x \)          |
| \( \cos x \)  | \( -\sin x \)         |
| \( \sec x \)  | \( \sec x \tan x \)   |
| \( \csc x \)  | \( -\csc x \cot x \)  |
| \( \tan x \)  | \( \sec^2 x \)        |
| \( \cot x \)  | \( -\csc^2 x \)       |
| \( \sin^{-1} x\) | \( \frac{1}{\sqrt{1-x^2}} \) \( |x| < 1 \) |
| \( \cos^{-1} x\) | \( \frac{-1}{\sqrt{1-x^2}} \) \( |x| < 1 \) |
| \( \tan^{-1} x\) | \( \frac{1}{1+x^2} \) \( x \in R \) |
| \( \cot^{-1} x\) | \( \frac{-1}{1+x^2} \) \( x \in R \) |
| \( \sec^{-1} x\) | \( \frac{1}{|x| \sqrt{x^2-1}} \) \( |x| > 1 \) |
| \( \csc^{-1} x\) | \( \frac{-1}{|x| \sqrt{x^2-1}} \) \( |x| > 1 \) |
(3) **Basic Theorems of Differentiation:**

1. \( \frac{d}{dx} (f(x) \pm g(x)) = f'(x) \pm g'(x) \)

2. \( \frac{d}{dx} (k(f(x))) = k \frac{d}{dx} f(x) \)

3. \( \frac{d}{dx} (f(x) \cdot g(x)) = f(x) g'(x) + g(x) f'(x) \)

4. \( \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \)

(4) **Different methods of differentiation:**

(4.1) **Differentiation of function of a function:**

If \( f(x) \) and \( g(x) \) are differentiable functions, then \( f(g(x)) \) is also differentiable and

\[
\frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x)
\]

is known as chain rule can be extended for two on move functions.

For example : \( \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{du} \cdot \frac{du}{dx} \).

**Note:**

- \( \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\left( \frac{dx}{dy} \right)} \)

- \( \frac{d^2 y}{dx^2} \neq \frac{1}{\left( \frac{d^2 x}{dy^2} \right)} \)

- If \( y = f(x) \) and \( x = g(y) \) are inverse functions of each other, then \( g(f(x)) = x \) and \( g'(f(x))f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{g'(x)} \)

\( \therefore g'(y)f'(x) = 1. \)

(4.2) **Implicit differentiation:**

If \( f(x, y) = 0 \) is an implicit function, then

\[
\frac{dy}{dx} = -\left( \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \right), \text{ where } \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ are partial differential coefficients of } f(x, y) \text{ with respect to } x \text{ and } y \text{ respectively.}
\]

**Note:**

Partial differential coefficient of \( f(x, y) \) with respect to \( x \) means the ordinary differential coefficient of \( f(x, y) \) with respect to \( x \) keeping \( y \) constant.
(4.3) **Logarithmic Differentiation**:

If function is of the form \( y = (f(x))^g(x) \) or \( y = \frac{f_1(x)f_2(x)\ldots}{g_1(x)g_2(x)\ldots} \), where \( g_i(x) \neq 0 \) and \( f(x), g(x) \) are differentiable functions \( \forall i = \{1, 2, 3, \ldots, n\} \), then differentiations after taking log of LHS and RHS is convenient and this method of differentiation is termed as logarithmic differentiation.

\[
\begin{align*}
\text{If } y &= (f(x))^g(x) \Rightarrow \log_e y = g(x) \cdot \log_e f(x) \\
\Rightarrow y &= e^{g(x) \cdot \log_e f(x)} \\
\therefore \frac{dy}{dx} &= e^{g(x) \cdot \log_e f(x)} \left( g'(x) \cdot \log_e f(x) + g(x) \cdot \frac{f'(x)}{f(x)} \right)
\end{align*}
\]

(4.4) **Parametric Differentiation**:

If \( x = f(\theta) \) and \( y = g(\theta) \), then

\[
\frac{dy}{dx} = \frac{dg}{df}.
\]

\[
\frac{dy}{dx} = \frac{g'(\theta)}{f'(\theta)} = \left( \frac{dy}{d\theta} \right) \left( \frac{d\theta}{dx} \right) = \frac{g'(\theta)}{f'(\theta)}.
\]

**Note:**

\[
\frac{d^2y}{dx^2} \neq \frac{g''(\theta)}{f''(\theta)}.
\]

(4.5) **Derivative of one function with respect to another**:

Let \( u = f(x) \) and \( v = g(x) \) be two functions of \( x \), then derivative of \( f(x) \) w.r.t. \( g(x) \) is given by:

\[
\frac{df(x)}{dg(x)} = \frac{f'(x)}{g'(x)}.
\]

(4.6) **Differentiation of infinite series**:

If \( y \) is given in the form of infinite series of \( x \) and we have to find out \( \frac{dy}{dx} \) then we remove one or more terms, it does not affect the series

(i) If \( y = \sqrt{f(x)} + \sqrt{f(x)} + \sqrt{f(x)} + \ldots \), then \( y = \sqrt{f(x)} + y \Rightarrow y^2 = f(x) + y \)

\[
2y \frac{dy}{dx} = f'(x) + \frac{dy}{dx}, \quad \therefore \frac{dy}{dx} = \frac{f'(x)}{2y - 1}
\]

(ii) If \( y = f(x)^y(x^y(x^y)^\ldots) \), then \( y = f(x)^y \)

\[
\frac{1}{y} \frac{dy}{dx} = \frac{yf'(x)}{f(x)} + \log f(x), \quad \therefore \frac{dy}{dx} = \frac{y^2f'(x)}{f(x)[1 - y \log f(x)]}
\]
(iii) If \( y = f(x) + \frac{1}{f(x)} \) then \( \frac{dy}{dx} = \frac{y f'(x)}{2y - f(x)} \)

(4.7) **Derivative Of Inverse Trigonometric Functions**:
Inverse trigonometric functions can be differentiation directly by use of standard results and the chain rule, but is is always easier by use of the proper substitution however the conditions of substitution must be applied carefully.
For example:

(4.8) **Differentiation using substitution**:
In case of differentiation of irrational function, method of substitution makes the calculations simpler and there the following substitution must be remember.

**Some suitable substitutions**

<table>
<thead>
<tr>
<th>S. N.</th>
<th>Functions</th>
<th>Substitutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( x = 0 \sin \theta ) or ( a \cos \theta )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( \sqrt{x^2 + a^2} )</td>
<td>( x = a \tan \theta ) or ( a \cot \theta )</td>
</tr>
<tr>
<td>(iii)</td>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \sec \theta ) or ( a \cosec \theta )</td>
</tr>
<tr>
<td>(iv)</td>
<td>( \sqrt{\frac{a-x}{a+x}} )</td>
<td>( x = a \cos 2\theta )</td>
</tr>
<tr>
<td>(v)</td>
<td>( \frac{a^2 - x^2}{a^2 + x^2} )</td>
<td>( x^2 = a^2 \cos 2\theta )</td>
</tr>
<tr>
<td>(vi)</td>
<td>( \sqrt{ax - x^2} )</td>
<td>( x = a \sin^2 \theta )</td>
</tr>
<tr>
<td>(vii)</td>
<td>( \frac{x}{a + x} )</td>
<td>( x = a \tan^2 \theta )</td>
</tr>
<tr>
<td>(viii)</td>
<td>( \frac{x}{a - x} )</td>
<td>( x = a \sin^2 \theta )</td>
</tr>
<tr>
<td>(ix)</td>
<td>( \sqrt{(x - a)(x - b)} )</td>
<td>( x = a \sec^2 0 - b \tan^2 \theta )</td>
</tr>
<tr>
<td>(x)</td>
<td>( \sqrt{(x - a)(b - x)} )</td>
<td>( x = a \cos^2 0 + b \sin^2 \theta )</td>
</tr>
</tbody>
</table>

(4.9) **Differentiation of determinant**:
If \( F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} \), where \( f, g, h, l, m, n, u, v, w \) are differentiable functions of \( x \) then \( F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l'(x) & m'(x) & n'(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix} \)

(4.10) **Differentiation of definite integral**:
If \( g_1(x) \) and \( g_2(x) \) both functions are defined on \([a, b]\) and differentiable at point \( x \in (a, b) \) and \( f(t) \) is continuous for \( g_1(a) \leq f(t) \leq g_2(b) \)
Then \( \frac{d}{dx} \int_{g_1}^{g_2} f(t) \, dt = f[g_2(x)]g'_2(x) - f[g_1(x)]g'_1(x) = f[g_2(x)] \frac{d}{dx} g_2(x) - f[g_1(x)] \frac{d}{dx} g_1(x) \).

(5) Derivatives of Higher Order:

Let a function \( y = f(x) \) be defined on an open interval \((a,b)\). Its derivative, if it exists on \((a,b)\) is a certain function \( f'(x) \) [or \((dy/dx)\) or \(y'\)] and is called the first derivative of \( y \) w.r.t. \( x \).

If it happens that the first derivative has a derivative on \((a,b)\) then this derivative is called the second derivative of \( y \) w.r.t. \( x \) & is denoted by \( f''(x) \) or \( \frac{d^2y}{dx^2} \) or \( y'' \).

Similarly, the 3rd order derivative of \( y \) w.r.t. \( x \), if it exists, is defined by \( \frac{d^3y}{dx^3} = \frac{d}{dx} \left( \frac{d^2y}{dx^2} \right) \). It is also denoted by \( f'''(x) \) or \( y''' \).
14. TANGENT AND NORMAL

(1) **Equation of Tangent and Normal** :

Let \( y = f(x) \) be the equation of a curve and point \( P(x_1, y_1) \) lies on the curve. For the curve \( y = f(x) \), geometrically \( f'(x_1) \) represents the slope to tangent to curve at location \( x = x_1 \), hence the equations of tangent can be obtained by point slope form of line.

\[
\frac{y - y_1}{x - x_1} = f'(x_1)
\]

is the equation of tangent at point \( P(x_1, y_1) \) to curve \( y = f(x) \).

Slope of normal to curve \( y = f(x) \) at point \( P(x_1, y_1) \) is given by

\[
\frac{-1}{f'(x_1)}
\]

is equation of normal at point \( P(x_1, y_1) \) to curve \( y = f(x) \).

**Note:**
If slope of tangent is zero, then tangent is termed as 'horizontal tangent' and if slope of tangent approaches infinite, then tangent is termed as 'vertical tangent'.

(2) **Length of Tangent and Normal** :

Let \( P(x_1, y_1) \) be any point on curve \( y = f(x) \). If tangent drawn at point \( P \) meets x-axis at \( T \) and normal at point \( P \) meets x-axis at \( N \), then the length \( PT \) is called the length of tangent and \( PN \) is called length of normal.

Projection of segment \( PT \) on x-axis (i.e. \( QT \)) is called the subtangent and similarly projection of line segment \( PN \) on x-axis (i.e. \( QN \)) is called subnormal. (refer figure (1)).

\[
\text{For curve } y = f(x): \\
P_T = \text{Length of tangent} \\
P_N = \text{Length of normal} \\
P_Q = |y_1|
\]

\[
\text{Figure (1)}
\]

Let \( m = \left( \frac{dy}{dx} \right)_P = \text{slope of tangent at } P(x_1, y_1) \text{ to curve } y = f(x) \).

\[
\therefore m = \tan \theta = \left( \frac{dy}{dx} \right)_P
\]

- In \( \Delta PTQ \), \( \sin \theta = \frac{y_1}{PT} \Rightarrow PT = |y_1 \csc \theta|

\[
\therefore \text{Length of tangent} = PT = \left| y_1 \sqrt{1 + \left( \frac{dx}{dy} \right)_P^2} \right|
\]
• In $\triangle PQN$, $\sin(90 - \theta) = \frac{y_1}{PN} \Rightarrow PN = |y_1 \sec \theta|$

$\therefore$ Length of normal $= PN = \left| y_1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right|

• In $\triangle PTQ$, $\tan \theta = \frac{y_1}{QT} \Rightarrow QT = |y_1 \cot \theta|

$\therefore$ Length of subtangent $= \left| \frac{y_1}{\frac{dy}{dx}} \right|

• In $\triangle PQN$, $\tan(90 - \theta) = \frac{y_1}{QN} \Rightarrow QN = |y_1 \tan \theta|

$\therefore$ Length of subnormal $= \left| y_1 \left(\frac{dy}{dx}\right)\right|

(3) **Angle between the curves:**

Angle between two intersecting curves is defined as the angle between their tangents or the normals at the point of intersection of two curves.

\[
\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}, \text{ where } \theta \text{ is acute angle of intersection and } (\pi - \theta) \text{ is obtuse angle of intersection.}
\]

**Note:**

- The curves must intersect for the angle between them to be defined.
- If the curves intersect at more than one point then angle between the curves is mentioned with references to the point of intersection.
- Two curves are said to be orthogonal if angle between them at each point of intersection is right angle (i.e. $m_1 m_2 = -1$).

(4) **Shortest distance between two curves:**

If shortest distance is defined between two non-intersecting curves, then it lies along the common normal.

\[
\text{Shortest distance between curves } y = f(x) \text{ and } y = g(x) \text{ is equal to } PQ.
\]
15. ROLLE'S THEOREM & MEAN VALUE THEOREM

(1) Rolle's Theorem:

If a function \( f(x) \) is
(i) continuous in a closed interval \([a, b]\)
(ii) derivable in the open interval \((a, b)\)
(iii) \( f(a) = f(b) \),
then there exists at least one real number \( c \) in \((a, b)\) such that \( f'(c) = 0 \)

Geometrical Illustration:

Let \( P \) be a point on the curve \( f(x) \) such that

\[
\frac{f(b) - f(a)}{b - a} = f'(c)
\]

The slope of chord AB is \( \frac{f(b) - f(a)}{b - a} \) and that of the tangent of point P is \( f'(c) \). These being equal, by (i), it follows that there exists a point P on the curve, the tangent at which is parallel to the chord AB.

(2) Langrange's Mean Value Theorem:

If a function \( f(x) \) is
(i) continuous in the closed interval \([a,b]\) and
(ii) derivable in the open interval \((a, b)\),
then there exists at least one real number \( c \) in \((a,b)\) such that

\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

Geometrical Illustration:
There may exist more than one point between A and B, the tangents at which are parallel to the chord AB. Lagrange's mean value theorem ensures the existence of at least one real number \( c \) in \( (a,b) \) such that
\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

(3) Alternative Form of Lagrange's Theorem:

If a function \( f(x) \) is
(i) continuous in a closed interval \([a, a + h]\)
(ii) differentiable in a open interval \((a, a + h)\), then there exists at least one number \( \theta \) lying between 0 and 1 such that
\[
f'(a + \theta h) = \frac{f(a + h) - f(a)}{h}
\]
i.e. \( f(a + h) = f(a) + hf'(a + \theta h) \) \((0 < \theta < 1)\)

(4) Physical Illustration:

\([f(b) - f(a)]\) is the change in the function \( f \) as \( x \) changes from \( a \) to \( b \) so that \( \frac{[f(b) - f(a)]}{b - a} \) is the average rate of change of the function over the interval \([a,b]\). Also \( f'(c) \) is the actual rate of change of the function for \( x = c \). Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval. i.e.

there exists a time \( \tau \in (t_1, t_2) \) for which
\[
x'(\tau) = \frac{x(t_2) - x(t_1)}{t_2 - t_1}
\]
i.e. instantaneous velocity = average velocity.

(5) Intermediate Value Theorem (IVT)

If a function \( f \) is continuous in a closed interval \([a,b]\) then there exists at least one \( c \in (a,b) \) such that
\[
f(c) = \frac{f(a) + \lambda f(b)}{1 + \lambda}
\]
for all \( \lambda \in \mathbb{R}^+ \).
16. MONOTONOCITY

Basic Definitions

(i) **Strictly increasing function:**
A function \( f(x) \) is said to be strictly increasing in an interval 'D', if for any two locations \( x = x_1 \) and \( x = x_2 \), where \( x_1 > x_2 \) ⇔ \( f(x_1) > f(x_2) \) \( \forall x_1, x_2 \in D \).
If function \( f(x) \) is differentiable in interval 'D', then for strictly increasing function \( f'(x) > 0 \). for example: \( f(x) = e^x \forall x \in R \)

(ii) **Strictly decreasing function:**
A function \( f(x) \) is said to be strictly decreasing in an interval 'D', if for any two locations \( x = x_1 \) and \( x = x_2 \), where \( x_1 > x_2 \) ⇔ \( f(x_1) < f(x_2) \) \( \forall x_1, x_2 \in D \).
If function \( f(x) \) is differentiable in interval 'D', then for strictly decreasing function \( f'(x) < 0 \). for example: \( f(x) = e^{-x} \forall x \in R \)

Note:
For strictly increasing or strictly decreasing functions, \( f'(x) \neq 0 \). For any point location or sub interval

A. Monotonocity about a point

1. A function \( f(x) \) is called an increasing function at point \( x = a \). If in a sufficiently small neighbourhood around \( x = a \).
   \[ f(a-h) < f(a) < f(a+h) \]

2. A function \( f(x) \) is called a decreasing function at point \( x = a \) if in a sufficiently small neighbourhood around \( x = a \).
   \[ f(a-h) > f(a) > f(a+h) \]

Note:
If \( x = a \) is a boundary point then use the appropriate one sided inequality to test monotonocity of \( f(x) \).
3. Test for increasing and decreasing functions at a point
   (i) If $f'(a) > 0$ then $f(x)$ is increasing at $x = a$.
   (ii) If $f'(a) < 0$ then $f(x)$ is decreasing at $x = a$.
   (iii) If $f'(a) = 0$ then examine the sign of $f'(a^+)$ and $f'(a^-)$
         (a) If $f'(a^+) > 0$ and $f'(a^-) > 0$ then increasing
         (b) If $f'(a^+) < 0$ and $f'(a^-) < 0$ then decreasing
         (c) otherwise neither increasing nor decreasing.

B. Monotonicity over an interval
1. A function $f(x)$ is said to be monotonically increasing for all such interval $(a, b)$ where
   $f'(x) \geq 0$ and equality may hold only for discrete values of $x$. i.e. $f'(x)$ does not identically
   become zero for $x \in (a, b)$ or any sub interval.
2. $f(x)$ is said to be monotonically decreasing for all such interval $(a, b)$ where $f'(x) \leq 0$ and
   equality may hold only for discrete values of $x$.{By discrete points, it mean that points
   where $f'(x) = 0$ don't form an interval}

Note:
   (i) A function is said to be monotonic if it's either increasing or decreasing.
   (ii) The points for which $f'(x)$ is equal to zero or doesn't exist are called critical points.
       Here it should also be noted that critical points are the interior points of an interval.
   (iii) The stationary points are the points where $f'(x) = 0$ in the domain.

C. Classification of functions
   Depending on the monotonic behaviour, functions can be classified into following cases.
   1. Increasing functions
   2. Non decreasing functions
   3. Decreasing functions
   4. Non-increasing functions

This classification is not complete and there may be function which cannot be classified into
any of the above cases for some interval $(a, b)$. 